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# *Dyadic Analysis of Space–Time Congruences*

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# Dyadic Analysis of Space–Time Congruences*

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A physical 3-vector and dyadic formalism for the treatment of general relativistic problems is derived, by systematic introduction of a proper tetrad field. The method is especially appropriate when there exists a physically or geometrically preferred timelike congruence; all quantities in the formalism are then shown to have immediate physical interpretation as proper local observables. A complete and nonredundant set of equations for the analysis of timelike congruences is developed in this operational language. Application is made to some simple examples involving local observations, and the direct measurement of the Riemann tensor discussed.

## A. INTRODUCTION

THE spinor analysis and the tetrad (or vierbein) formalism were both employed in the 1930's, in connection with attempts to generalize general relativity and to formulate a unified theory of electricity and gravitation. The lack of success in this particular endeavor, however, led to a subsequent lack of interest in the formal techniques thus opened up. Now, in just the last few years, greatly renewed interest in the spinor analysis has followed upon its successful application to cases of gravitational radiation, *within* the now-classical theory of Einsteinian general relativity.[1] Such cases are characterized by having preferred null congruences. The tetrad formalism, we believe, can be of equally great service, within Einstein theory, when appropriately applied to situations having preferred timelike congruences. When a tetrad formalism is based on a preferred congruence it then naturally leads to a three-dimensional dyadic and vector formulation which explicitly depends on (and expresses) the dimensionality and signature of physical space–time. For the many important results that depend on this dimensionality and signature for their validity, the usual tensor calculus is rather an imperfect instrument, tending to prove easily only more general results, valid in $n$ dimensions with arbitrary signatures.

The dyadic formalism we present in the present paper has the advantages of physical *interpretability*, mathematical *completeness*, and wide *applicability*. We are at considerable pains in several sections of the paper to give the physical interpretation of all dyadic quantities arising from the formalism—in almost all cases this is rather easily done, for indeed the naturally occurring dyadic quantities are found to be those already familiar either from classical mechanics or from quite simple geometric considerations. The result is a much more understandable set of relations, than in the more customary 4-tensor formulation of general relativity, especially when a physically distinguished congruence is present. The second advantage is in the completeness of the dyadic partial differential equations. The more usual tensorial techniques for discussing congruences in curved $(3 + 1)$-dimensional manifolds are quite *ad hoc*, and although the literature is replete with many elegant results for special cases, a systematic mathematical approach or algorithm which overlooks no such results, writes no redundant equations, and yet is completely general, seems not to be available. Although this technical point is difficult to express in an introduction, it should become clear in the body of the paper. Finally, there promise to be many areas of application of the dyadic formalism: a timelike congruence which is in some way distinguished or preferred occurs in such varied situations as space–times supporting matter–energy distributions, cosmological models with preferred galactic distributions, and space–times having symmetries and isometries described by congruences. The possibility of generating new exact solutions of the field equations should also be mentioned, especially since the dyadic formalism is not wedded to a choice of (holonomic) coordinates. The applicability of the dyadic formalism to the explicit prediction of experimental results is noteworthy: the dyadic quantities are world scalars, proper components everywhere resolved along the orthogonal space and time axes of local Lorentz tetrads; they are, that is, precisely the raw material of observational physics. We demonstrate this last point by presenting equations for the differential absolute acceleration and precession between adjacent inertially oriented test particles, which show in principle how 14 components of the Riemann tensor are locally measurable.

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[1] E. Newman and R. Penrose, J. Math Phys. 3, 566 (1962).

The differential precession equation in particular seems not to have been given previously in a form involving strictly local, proper, observations, and uniting the differential Thomas precession of accelerating particles with the general relativistic Fokker precession.

In Sec. B of this paper we discuss tetrad fields and the formulation of general relativity in terms of such anholonomic reference systems. Section C introduces the 3-dyadic formalism, based on a tetrad field attached to a preferred timelike congruence, and elucidates the physical significance of the dyadic quantities. In Sec. D we transcribe the tetrad equations into this physical dyadic language.

In a succeeding paper[2] we will demonstrate the utility of the dyadic formalism in a consideration of the Herglotz–Nöther theorem on the motion of Born-rigid bodies, which assumes a simplicity otherwise entirely concealed. In addition we will there derive new results for Born congruences in curved space–times. In future papers, we intend to present the dyadic method applied in several other situations having, again, preferred timelike congruences.

## B. TETRAD FORMALISM

### 1. Tetrad Fields

The use of auxiliary ennuples in differential geometry is of course not new, going back at least to the work of Ricci. To introduce the 3-dyadic treatment of Secs. C and D, we nevertheless must briefly recapitulate in a uniform notation much of the formalism expounded, for example, in Schouten[3] and Eisenhart.[4]

The method of analysis follows upon systematic introduction of a tetrad field based on a given timelike congruence; we will in fact use four orthonormal reference vector fields ${}_r\lambda^\mu$, where $r = 0$ labels a timelike vector, and $r = 1, 2, 3$ are three spacelike vectors. The label $r$ is a "Lorentz index" in the terminology of Synge,[5] and we will reserve Latin indices for this purpose. These unit vector fields ${}_r\lambda^\mu$ will trace out four congruences not, in general, 3-surface orthogonal. The method is thus equivalent to the introduction of convenient, everywhere orthogonal, but anholonomic coordinates, in the terminology of Schouten.[3]

[2] H. D. Wahlquist and F. B. Estabrook, unpublished.

[3] J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, 1954), 2nd ed.

[4] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1926).

[5] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960).

By transvection with the contravariant tetrad vectors ${}_r\lambda^\mu$ or their covariant duals, ${}^s\lambda_\nu$, we will systematically "strangle" all tensor indices of fields of interest, thus replacing these indices by Lorentz indices, labeling the resulting arrays of world scalars. This formalism in many ways bridges the conventional approach in which tensors are considered as arrays of components, and that of the school of Cartan, with its perhaps more physical emphasis on algebraic quantities in tangent vector spaces.[6]

At any point of space–time, the given timelike congruence, and in particular the orthonormal vector tetrad there, defines a preferred local Minkowskian frame, with respect to which Lorentz indices take meaning as labeling proper components, spacelike, timelike, and mixed. We will use the special relativistic Minkowski metric form $\eta^{rs} = \eta_{rs} = \mathrm{diag}(-1, 1, 1, 1)$ to raise and lower Lorentz indices, and so to express the tetrad orthonormality relations

$$ {}^r\lambda^\mu\, {}^s\lambda_\mu = \eta^{rs}, \quad {}_r\lambda^\mu\, {}_s\lambda_\mu = \eta_{rs}. \tag{B.1} $$

The metric tensor components are, as in the Cartan formalism, simply given by quadratic forms in the unit vectors:

$$ \begin{gathered} g_{\mu\nu} = {}^r\lambda_\mu\, {}_r\lambda_\nu, \quad g^{\mu\nu} = {}^r\lambda^\mu\, {}_r\lambda^\nu, \\ g^\mu_\nu \equiv \delta^\mu_\nu = {}^r\lambda^\mu\, {}_r\lambda_\nu. \end{gathered} \tag{B.2} $$

In general, it appears that results which are valid *only* for a certain dimensionality and signature of a space, are much more easily and directly demonstrated with such a tetrad formalism. The main algebraic inconvenience which will arise is due to the lack of commutivity in the process of successive "intrinsic" differentiation of scalars (i.e., absolute differentiation along the unit vector fields); we derive the necessary commutation formulas in Sec. B3.

### 2. The Object of Anholonomity

The variation of the tetrad field is described by the set of strangled intrinsic derivatives of the unit vectors:

$$ \Gamma_{rst} \equiv {}_t\lambda^\mu\, {}_s\lambda_{\mu;\nu}\, {}_r\lambda^\nu. \tag{B.3} $$

These are essentially the "rotation coefficients" introduced by Ricci. It is shown in Sec. B3 that the set of scalars $\Gamma_{rs}{}^t$ can properly be regarded as the anholonomic components of the affinity in our 3 + 1 metric space. From Eq. (B.1) it immediately follows that $\Gamma_{rst} = -\Gamma_{rts}$ and indeed there are here exactly 24 scalar fields. A more elegant set of 24 scalars,

[6] See, for example, A. Lichnerowicz, *Elements of Tensor Calculus* (Methuen and Company, Ltd., London, 1962).

however, may be defined using only simple curls of the vector fields:

$$\Omega^{r}_{\cdot st} \equiv \tfrac{1}{2}({}^{r}\lambda_{\mu,\nu} - {}^{r}\lambda_{\nu,\mu})_{t}\lambda^{\mu}{}_{s}\lambda^{\nu}. \tag{B.4}$$

In our metric space this is equivalent to

$$\Omega_{rst} = \tfrac{1}{2}(\Gamma_{tsr} - \Gamma_{str}). \tag{B.5}$$

The $\Omega_{tsr}$ are again antisymmetric on the last pair of indices: $\Omega_{tsr} = -\Omega_{trs}$. For the present case of orthonormal unit vectors the Eq. (B.5) can be readily solved for the anholonomic affinity components, which demonstrates the equivalence of the two sets:

$$\Gamma_{tsr} = \Omega_{str} + \Omega_{rst} + \Omega_{tsr}. \tag{B.6}$$

It is thus clear that the curls of the tetrad fields carry all the metric information, and so knowledge of the 40 Christoffel symbols is not now required. This is an advantage of an orthonormal tetrad formulation, also met with in the spinor calculus, where there are just 24 components in the spin connection. In the following we give explicit expressions for the Riemann tensor components in terms of the $\Gamma_{tsr}$ fields.

The components $\Omega^{r}_{\cdot st}$ defined as in Eq. (B.4) are termed the "object of anholonomity" by Schouten,[3] who introduces them in general, non-Riemannian, spaces. The vanishing of the $\Omega^{r}_{\cdot st}$ everywhere is the integrability condition for the unit vectors to be gradients of four families of hypersurfaces—hence, derivable from ordinary or holonomic coordinates. In our present case, the vanishing of $\Omega^{r}_{\cdot st}$ would imply the existence of four everywhere orthogonal, equally spaced (hence, Cartesian) coordinate families, which is to say, the flatness of space–time.

Intrinsic differentiation of Eq. (B.4) with respect to ${}_{p}\lambda^{\sigma}$ and subsequent complete antisymmetrization with respect to Lorentz indices $s$, $t$, and $p$, results in a set of 16 first-order differential identities:

$$\Omega^{r}_{\cdot[st,p]} = 2\Omega^{q}_{\cdot[ps}\Omega^{r}_{\cdot t]q}. \tag{B.7}$$

Here the brackets denote complete antisymmetrization—in the case of three indices, this involves adding six terms with appropriate signs according to the even or odd permutation of the indices, and multiplication by $\frac{1}{6}$. These equations are to be found in Ref. 3, p. 101; they are in fact *integrability* conditions on the 24 world scalar fields $\Omega^{r}_{\cdot st}$, allowing them to be derivable from four congruences or vector fields ${}_{s}\lambda^{\mu}$ in the manner given.

The 16 integrability conditions are especially noteworthy, in that the metric properties of space–time nowhere enter in their derivation. There are twenty other equations implied in a metric space–time by the form of Eq. (B.5); when second covariant derivatives are eliminated by antisymmetrization (this time on two indices only) components of the strangled Riemann tensor $R^{rstp}$ are introduced. If the 16 relations already written are systematically eliminated, by imposing the algebraic symmetries of the Riemann tensor in metric 4-space, one finally obtains the further independent set:

$$\begin{aligned}\Omega^{(rp)(s,t)} + \Omega^{(st)(r,p)} &= 2\Omega^{(rp)q}\Omega^{(st)}_{\cdot\cdot q} - \Omega^{(sp)q}\Omega^{(rt)}_{\cdot\cdot q} \\ &\quad - \Omega^{(rs)q}\Omega^{(tp)}_{\cdot\cdot q} - \tfrac{3}{2}\Omega^{q((t(p}\Omega^{\cdot r)s)}_{q} \\ &\quad + \Omega^{q(p(t}\Omega^{s)r)}_{\cdot\cdot q} + \Omega^{q((t(p}\Omega^{r)s)}_{\cdot\cdot q} - \tfrac{3}{4}S^{strp}.\end{aligned} \tag{B.8}$$

Here we have used parentheses to denote total symmetrization—in the case of two indices, for example, this means summation of two terms with indices transposed, and multiplication by $\frac{1}{2}$. In addition, it has proved algebraically convenient to use the *symmetrized* Riemann tensor (Ref. 5, p. 54),

$$S^{strp} = -\tfrac{1}{3}(R^{srtp} + R^{sptr}). \tag{B.9}$$

It is clear that all of Eq. (B.8) has the same symmetries as $S^{strp}$: viz., symmetry on the first pair of indices, symmetry on the second pair, symmetry on the two pairs of indices, and a cyclic symmetry on, say, the last three indices. Hence there are precisely 20 independent relations in Eq. (B.8). The complete set of 36 differential relations for the tetrad field, consists of Eqs. (B.7) and (B.8).

Although their separate origins are obscured by the process, it is often convenient to have Eqs. (B.7) and (B.8) written together in one set of 36 equations involving the usual Riemann tensor, the anholonomic affinity components, and their intrinsic derivatives (Ref. 4, p. 98):

$$\begin{aligned}\Gamma^{[t|sr|,p]} &= \tfrac{1}{2}\Gamma^{prq}\Gamma^{ts}_{\cdot\cdot q} - \tfrac{1}{2}\Gamma^{trq}\Gamma^{ps}_{\cdot\cdot q} \\ &\quad + \Gamma^{[pt]q}\Gamma^{\cdot sr}_{q} + \tfrac{1}{2}R^{srtp},\end{aligned} \tag{B.10}$$

where indices enclosed between bars are excluded from the antisymmetrization brackets. Equation (B.10) is, of course, also the promised direct expression of the components of the Riemann tensor in terms of the tetrad field.

### 3. Further Relations

In Sec. D the dyadic forms of Eqs. (B.7) and (B.8) are presented as a general tool for the analysis of space–time congruences. We must, however, first supplement these equations by commutation formulas, and by the Bianchi Identities.

Because of the anholonomity, two successive intrinsic derivative indices do not commute—even though they are derivatives of world scalars. This is easily seen from the definition of intrinsic derivative; it is perhaps more illuminating, however, to derive the important resulting commutation formula from the general formula for strangulation of covariant derivatives. Consider a tensor $T^{\tau\cdot\cdot}_{\cdots\mu\cdots;\sigma}$ with a single covariant differentiation index; strangle by multiplication with ${}^t\lambda_\tau \cdots {}_u\lambda^\mu \cdots {}_s\lambda^\sigma$; using the orthonormal properties of the tetrad, the resulting expression can be written in terms of intrinsic derivatives:

$$(T^{\tau\cdot\cdot}_{\cdots\mu\cdots;\sigma})\,{}^t\lambda_\tau \cdots {}_u\lambda^\mu \cdots {}_s\lambda^\sigma = T^{t\cdot\cdot}_{\cdots u,s} + \Gamma_{sq}^{\cdot\cdot t}T^{q\cdot\cdot}_{\cdots u\cdot\cdot} + \cdots - \Gamma_{su}^{\cdot\cdot q}T^{t\cdot\cdot}_{\cdots q\cdot\cdot} - \cdots . \qquad \text{(B.11)}$$

In this scalar expression the set of $\Gamma_{st}^{\cdot\cdot r}$ plays exactly the formal role of an affinity. We emphasize, however, that whereas with ordinary holonomic coordinates an affinity in a Riemann space is symmetric on the first two indices (and so in four dimensions has 40 components), as a result of the orthonormal nature of our present anholonomic reference frame $\Gamma_{st}^{\cdot\cdot r}$ is antisymmetric on the last two indices and in four dimensions has 24 components.[7]

Since we may commute covariant derivatives of any scalar, $T^{t\cdot\cdot}_{\cdots u\cdots;[\nu\sigma]} = 0$, it then follows immediately upon strangulation according to Eq. (B.11) that the commutation formula for intrinsic differentiation is (suppressing all nonderivative Lorentz indices)

$$T_{,[rs]} = \Gamma_{[sr]}^{\cdot\cdot q}T_{,q} = \Omega^q_{\cdot rs}T_{,q}. \qquad \text{(B.12)}$$

We conclude this section by recording the integrability conditions for the (20) components of the Riemann tensor field, allowing them to be derivable as in Eq. (B.8). If we are *given* a Riemannian metric form, these conditions are of course identically satisfied: they are indeed the Identities of Bianchi. In our tetrad notation, they follow readily upon intrinsic differentiation of Eq. (B.8), antisymmetrization, and use of the commutation relation Eq. (B.12) to eliminate second derivatives. The Bianchi Identities may be most easily written in terms of the strangled double-dual of the Riemann tensor; they are

$$*R^{prst}_{\cdots\cdots,t} + 2*R^{sqt[r}\Gamma_{qt}^{\cdot\cdot p]} + 2*R^{prt[q}\Gamma_{qt}^{\cdot\cdot s]} = 0, \qquad \text{(B.13)}$$

where

$$*R^{prst} = \tfrac{1}{4}\epsilon^{prqn}\epsilon^{stvu}R_{qnvu}, \qquad \text{(B.14)}$$

and $\epsilon^{prqn}$ is the usual four-dimensional permutation symbol. As is immediately obvious in the dyadic notation, there are exactly 20 independent conditions in Eq. (B.13); these include the four contracted Bianchi Identities. These 20 equations are of great importance and utility when deriving the consequences of special assumptions and symmetries imposed on the gravitational field, and on the stress-energy tensor; both of these, in Einstein's theory, are comprised in the geometrical Riemann tensor.

[7] It is mnemonically most convenient to write *all* the "correction" terms in Eq. (B.11) with plus signs, summing always on the *second* index of the anholonomic affinity.

## C. 3-VECTOR AND 3-DYADIC ALGEBRAIC FORMALISM AND INTERPRETATION

### 1. Introduction

In the general tetrad formalism the associated congruences are geometrical reference objects more or less devoid of intrinsic physical significance. If, however, we identify the timelike congruence with the world lines of a material continuum, described by the velocity 4-vector field ${}_0\lambda^\mu$, this is no longer the case. The ${}_0\lambda^\mu$ congruence might represent, in various instances, the motion of a relativistic fluid, a rigid body as defined by Born's constraint condition, a proper frame of reference for the performance of experiments, or a privileged cosmological matter distribution. But regardless of the particulars, it is the attitude of considering the timelike congruence to be a physically given object that provides the rationale for the 3-dyadic formalism to be presented here. A region of space–time in which such a congruence exists is endowed with a unique time direction at each point, and it becomes physically reasonable then to dissolve the 4-dimensional union of space and time with respect to the congruence. Of course, such a decoupling is almost always done at some point in any physical problem in relativity theory, by selection of a "convenient" set of coordinates. With the tetrad and dyadic formalisms this is done at the outset before further specification of the particular system at hand, and without prejudice as to the admissibility or desirability of any holonomic coordinate system.

In Sec. C 2 we introduce a representation of the anholonomic affinity, $\Gamma_{rs}^{\cdot\cdot t}$, by splitting its components into independent three-dimensional arrays having spacelike Lorentz indices only. The three spacelike tetrad vectors used to generate these components are not, of course, unique. In Sec. C3 it is shown that certain restricted transformations between sets of these auxiliary vectors are the analogs of the familiar orthogonal rotations of Cartesian axes in 3-space, and that the arrays of proper components will transform precisely as conventional 3-vector or 3-dyadic fields under such spatial rotations. A detailed discussion of the kinematical and geometrical significance of the quantities thus introduced is given in Secs. C4 and C5.

### 2. Three-Dimensional Representation of $\Gamma_{rst}$

The splitting apart of the components of the anholonomic affinity into independent 3-dimensional arrays proceeds by segregating those components which differ in the number and location of timelike indices, here denoted by 0. It should be noted that raising or lowering the 0 index changes the sign of a quantity. We shall henceforth reserve the letters from the first half of the Latin alphabet ($a \cdots m$) to indicate spacelike indices. These take on the values 1, 2, 3 only, and *the summation convention for such indices is limited* to this range. Since the local spacelike metric $\eta^{ab} = \delta^a_b$, the vertical position of these indices does not matter. Parentheses and brackets around indices have the same significance as in Sec. B, and $\epsilon_{abc}$ denotes the usual three-dimensional permutation symbol.

With these conventions, the components of $\Gamma_{rst}$ having at least one timelike index may be written:

$$\Gamma_{00a} = -\Gamma_{0a0} \equiv a_a, \tag{C.1}$$

$$-\Gamma_{ab0} = \Gamma_{a0b} \equiv S_{ab} + \epsilon_{abc}\Omega^c, \tag{C.2}$$

$$\Gamma_{0ab} = -\Gamma_{0ba} \equiv \epsilon_{abc}\omega^c, \tag{C.3}$$

where the quantities on the right constitute a three-dimensional representation consistent with the antisymmetry of $\Gamma_{rst}$ on its last two indices. The array of scalars, $S_{ab}$, is defined to be symmetric to the interchange of $a$ and $b$; from Eq. (C.2) it follows that

$$S_{ab} \equiv -\Gamma_{(ab)0}. \tag{C.4}$$

These definitions provide a representation for 15 of the 24 independent components of the affinity. The remaining nine, comprised in the wholly spacelike $\Gamma_{abc}$, describe characteristics of the nonunique auxiliary congruences. Again by virtue of the antisymmetry on $b$ and $c$, we may represent six of these quantities by a symmetric array, $N_{ad}$, and the final three by $L_b$ as follows:

$$\tfrac{1}{2}\epsilon_{dcb}\Gamma_a^{\cdot bc} \equiv N_{ad} - \tfrac{1}{2}N^b_{\cdot b}\delta_{ad} + \epsilon_{ad\cdot}^{\cdot\cdot b}L_b, \tag{C.5}$$

where the contraction, $N^b_{\cdot b}$, has been explicitly subtracted for reasons of formal simplicity later. From this equation we further have:

$$N_{ad} - \tfrac{1}{2}N^b_{\cdot b}\delta_{ad} \equiv \tfrac{1}{2}\epsilon_{(d}^{\cdot cb}\Gamma_{a)bc}, \tag{C.6}$$

$$N^b_{\cdot b} \equiv \epsilon_{abc}\Gamma^{abc}, \tag{C.7}$$

and

$$L_b \equiv \tfrac{1}{2}\Gamma^a_{\cdot ab}. \tag{C.8}$$

For future reference it is convenient also to catalog the components of the object of anholonomity, $\Omega_{rst}$, in terms of this representation, *viz.*,

$$\Omega_{00a} = -\Omega_{0a0} = \tfrac{1}{2}a_a, \tag{C.9}$$

$$\Omega_{0ab} = -\Omega_{0ba} = \epsilon_{abc}\Omega^c, \tag{C.10}$$

$$\Omega_{ab0} = -\Omega_{a0b} = \tfrac{1}{2}[-S_{ab} + \epsilon_{abc}(\Omega^c - \omega^c)], \tag{C.11}$$

$$\Omega_{abc} = -\Omega_{acb} = \tfrac{1}{2}(\epsilon^d_{\cdot cb}N_{ad} + 2L_{[c}\delta_{b]a}). \tag{C.12}$$

### 3. Vector-Dyadic Notation

In the representation just developed, the set of 24 components of either $\Gamma_{rst}$ or $\Omega_{rst}$ clearly falls into natural three-dimensional subarrays for which a vector and dyadic notation would be convenient. In such notation the equations involving these quantities would preserve the familiar formalism of 3-space rotation covariance which here corresponds to the arbitrariness remaining in the selection of the auxiliary spacelike tetrad vectors, even when ${}_0\lambda^\mu$ is physically given. Since the quantities in question are defined in terms of the tetrad vectors themselves and their derivatives, it is not obvious that this program must succeed at all; especially if we insist that the vector or dyadic character shall hold not just at a single event, but throughout space–time.

Accordingly, we now perform an analysis of the transformation properties of the arrays under a general, four-dimensional, proper orthogonal transformation of the tetrad fields which leaves ${}_0\lambda^\mu$ fixed. We determine the widest group of such transformations under which the arrays will have the 3-vector and dyadic character at every point. Not surprisingly, the set of acceptable transformations is quite restricted, in the sense that the parameters of the transformation at one event determine the transformation throughout space–time. For such transformations, however, we show that the arrays $a_a$ and $L_a$ are polar 3-vectors, say **a** and **L**; while $\Omega_a$ and $\omega_a$ form axial vectors, $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$. The symmetric arrays $S_{ab}$ and $N_{ab}$ transform as dyadics, **S** and **N**, although the latter has a pseudocharacter under inversions of the spatial tetrad vectors.

Consider then an orthogonal transformation of the three auxiliary spacelike vector fields. We may write such a transformation most generally as

$${}_a\bar{\lambda}^\mu = A^\mu_{\cdot\nu}\,{}_a\lambda^\nu, \tag{C.13}$$

where $A^\mu_{\cdot\nu}$ is a tensor field satisfying

$$A_{\mu\sigma}\,g^{\sigma\tau}A_{\nu\tau} = g_{\mu\nu}. \tag{C.14}$$

In the present case we require that the orthogonal tensor field be proper, and that it leave unchanged the ${}_0\lambda^\mu$ congruence; it follows that it has an unmoved 2-flat and can be written in the canonical form[8]

[8] F. B. Estabrook, California Institute of Technology, Pasadena, California, Jet Propulsion Laboratory, Research Summary No. 36-14, p. 119 (1962).

$$A_{\mu\nu} = \cos\theta\, g_{\mu\nu} + \sin\theta\, (-g)^{-\frac{1}{2}} \epsilon_{\mu\nu\sigma\tau}\, {}_0\lambda^\sigma \zeta^\tau + 2\sin^2(\tfrac{1}{2}\theta)\,(\zeta_\mu\zeta_\nu - {}_0\lambda_\mu\, {}_0\lambda_\nu). \quad (C.15)$$

$\zeta^\nu$ is a unit spacelike four-vector orthogonal to ${}_0\lambda^\mu$; together they define the unmoved 2-flat. In the local tetrad frame, we see a simple spatial rotation by angle $\theta$ about the $\zeta^\nu$ direction. Equation (C.14) and the invariance of ${}_0\lambda^\mu$ and $\zeta^\nu$ can be verified immediately by direct computation. Strangling Eq. (C.15) we get the familiar 3-space proper rotation matrix

$$O_{ab} = \cos\theta\, \delta_{ab} + \sin\theta\, \epsilon_{abc}\zeta^c + 2\sin^2(\tfrac{1}{2}\theta)\, \zeta_a\zeta_b. \quad (C.16)$$

$\boldsymbol{\zeta}$ is the unit 3-vector with strangled components $\zeta_a \equiv \zeta_\mu\, {}_a\lambda^\mu$; it points along the axis of the rotation. Noting that $O_{0a} = 0$, $O_{00} = -1$, we can also write

$$A_{\mu\nu} = O_{st}\, {}^s\lambda_\mu\, {}^t\lambda_\nu = O_{ab}\, {}^a\lambda_\mu\, {}^b\lambda_\nu - {}^0\lambda_\mu\, {}^0\lambda_\nu. \quad (C.17)$$

Any vector $V^\mu$ orthogonal to ${}_0\lambda^\mu$ may be expanded in either auxiliary tetrad system,

$$V^\mu = {}^aV\, {}_a\lambda^\mu = {}^b\bar{V}\, {}_b\bar{\lambda}^\mu, \quad (C.18)$$

and substituting from Eq. (C.13) we can see that the components ${}^aV$ transform contragradiently to the unit vectors:

$${}^c\bar{V} = {}^aVO_{a}^{\cdot c}. \quad (C.19)$$

The arrays of components $a_a$, $\Omega_a$ and $S_{ab}$ can be immediately shown, from their definitions Eqs. (C.1) and (C.2), to transform according to Eq. (C.19) (or its dyadic generalization, in the case of $S_{ab}$), and so this justifies our use of 3-vector and dyadic notation for them: **a**, $\boldsymbol{\Omega}$, and **S**.

We now consider the change of $\omega_c$, defined in Eq. (C.3), under the transformation of Eq. (C.13). From the definition,

$$\Gamma_{0ba} = -\epsilon_{abc}\omega^c. \quad (C.20)$$

If we similarly set

$$\bar{\Gamma}_{0ba} = {}_a\bar{\lambda}^\mu\, {}_b\bar{\lambda}_{\mu;\nu}\, {}_0\lambda^\nu = -\epsilon_{abc}\bar{\omega}^c, \quad (C.21)$$

substitution from Eqs. (C.13) and (C.16) leads finally to the transformation law

$$\bar{\omega}^a = \omega^d O_{d}^{\cdot a} + \tfrac{1}{2}\epsilon^{abc}\dot{O}^{d}_{\cdot b}O_{dc}. \quad (C.22)$$

Equivalent to this is

$$\bar{\omega}^a = \omega^d O_{d}^{\cdot a} - \dot{\theta}\zeta^a - \sin\theta\, \dot{\zeta}^a - (1-\cos\theta)\,(\boldsymbol{\zeta}\times\dot{\boldsymbol{\zeta}})^a, \quad (C.23)$$

where the superimposed dot means the intrinsic derivative in the ${}_0\lambda^\mu$ direction, e.g., $\dot{\theta} = \theta_{,\mu}\, {}_0\lambda^\mu$. If (and only if) we restrict the orthogonal transformation tensor $A^{\mu}_{\cdot\nu}$ to one for which $\dot{\theta}$ and $\dot{\boldsymbol{\zeta}}$ everywhere vanish, which is to say $\dot{O}_{ab} = 0$, we arrange that the quantities $\omega^a$ transform precisely like a 3-vector, and so justify our choice of notation for this set of three components. The restriction $\dot{O}_{ab} = 0$ amounts to correlating the rotation induced by $O_{ab}$ of the three spacelike unit vectors of a fundamental tetrad at a given event, to the rotations of all other such tetrads induced at all other events along the world line of the ${}_0\lambda^\mu$ congruence through the given event, so that $\boldsymbol{\omega}$ is not intrinsically changed, but only locally projected on a different anholonomic coordinate mesh.

The remaining components, those of the symmetric 3-dyadic $N_{ab}$ or **N**, and the 3-vector $L_a$, or **L**, will also transform precisely as the notation suggests only under special forms of $O_{ab}$. In fact, one finds

$$\bar{N}^{ab} = N^{cd}O_{c}^{\cdot a}O_{d}^{\cdot b} + \tfrac{1}{2}\epsilon^{hfg}O_{fd,g}O_{h}^{\cdot d}\delta^{ab} + \tfrac{1}{2}O_{fd,g}O^{g(a}\epsilon^{b)cd}O^{f}_{\cdot c} \quad (C.24)$$

and

$$\bar{L}^a = L^dO_{d}^{\cdot a} - \tfrac{1}{2}O^{ca}_{\cdot\cdot,c}. \quad (C.25)$$

Upon substitution of the explicit form of $O_{ab}$ from Eq. (C.16), it is found from equations analogous to Eq. (C.23) that the extraneous terms in Eqs. (C.24) and (C.25) can vanish in general if and only if $O_{ab,c} = 0$. Combining this with our previous result, we can state: $\boldsymbol{\omega}$, **N**, and **L** transform properly as three-dimensional vector and dyadic fields, for those orthogonal transformations having the array $O_{ab}$ constant everywhere.

We have then the following situation: *given* ${}_0\lambda^\mu$, a further orthonormal set ${}_a\lambda^\mu$ may be chosen at every event. Three quite arbitrary auxiliary spacelike congruences are thus determined. From this auxiliary set, however, we usually allow only transformations to other sets derived from it by choosing an arbitrary unit spacelike 3-vector $\boldsymbol{\zeta}$, *whose components with respect to the spacelike unit vectors are the same at every event*, and rotating the spacelike set at every event by the *same* angle $\theta$ about the direction $\boldsymbol{\zeta}$. Any such transformation thus derives a new set of three auxiliary spacelike congruences from the first. We call such a new set of auxiliary orthogonal congruences *3-space rotated* with respect to the original set. Under such 3-space rotation, **a**, $\boldsymbol{\Omega}$, **S**, $\boldsymbol{\omega}$, **N**, and **L** transform in familiar three-dimensional orthogonal fashion, and form-invariant equations between these quantities can be written in the familiar language of the Gibbsian vector analysis.

In such equations, **I** denotes the unit dyadic, with components $I_{ab} = \delta_{ab}$. By (tr **S**) we mean the contraction or trace, $S^{a}_{\cdot a}$. The dot notation for inner products is used, and a double dot product of two dyadics is equivalent to the trace of their inner product. The cross product is defined in the usual right-handed way. When a × operates on a dyadic, it operates on the nearest index when expressed in terms of components; e.g.,

$$(\mathbf{\Omega}\times\mathsf{S})_{ab} \equiv \epsilon_{acd}\Omega^c S^d_{\cdot b}. \qquad (C.26)$$

The double cross product of two dyadics often provides a convenient brevity of notation. It is used only between symmetric dyadics so that no ambiguity of ordering can arise in its definition; viz.,

$$(\mathsf{Q} \overset{\times}{\times} \mathsf{S})_{ab} \equiv \epsilon_{acd}\epsilon_{bfg}Q^{fc}S^{dg}. \qquad (C.27)$$

The result is again a symmetric dyadic having the expansion

$$\mathsf{Q}\overset{\times}{\times}\mathsf{S} \equiv \mathsf{Q}\cdot\mathsf{S} + \mathsf{S}\cdot\mathsf{Q} - (\mathrm{tr}\,\mathsf{S})\mathsf{Q} - (\mathrm{tr}\,\mathsf{Q})\mathsf{S} + [(\mathrm{tr}\,\mathsf{S})(\mathrm{tr}\,\mathsf{Q}) - \mathsf{Q}:\mathsf{S}]\mathsf{I}. \qquad (C.28)$$

We use the 3-vector symbol **D** for spatial intrinsic derivation: thus $\phi_{.a}$ becomes $\mathbf{D}\phi$, a gradient; $V^a_{\cdot\cdot a}$ becomes $\mathbf{D}\cdot\mathbf{V}$, a divergence; $\frac{1}{2}(V_{c,b} - V_{b,c})$ when multiplied by $\epsilon^{abc}$ becomes the curl, $\mathbf{D}\times\mathbf{V}$; etc. Another spatial differential operator, linearly related to **D**, is introduced in Sec. C5; denoted $\nabla$, this operator is convenient in many of our equations, and is the triad-strangled operation of covariant differentiation in spatial subspaces (when such exist). The operations of gradient, divergence and curl with the $\nabla$ operator are defined in Sec. C5.

### 4. Physical Interpretation of the Dyadic Quantities

The identification of ${}_0\lambda^\mu$ with a physical motion imbues many of the components of the anholonomic affinity with immediate physical or kinematical significance. We first develop the interpretations by recalling some definitions met with in the usual tensorial description of the kinematics of a relativistic continuum. In a sense this procedure is logically inverted, but it has the advantage of quickly connecting quantities in the present notation with the familiar tensor quantities. A more basic approach will follow.

Let a fluid motion be described by a velocity 4-vector field $\lambda^\mu$, with $\lambda_\mu\lambda^\mu = -1$. From the derivatives $\lambda^\mu_{\cdot;\nu}$ one resolves canonical sets of first-order differential quantities:[9] the acceleration vector $a_\mu \equiv \lambda_{\mu;\nu}\lambda^\nu$; the (antisymmetric) angular velocity tensor $\Omega_{\mu\nu} \equiv \lambda_{[\mu;\nu]} + a_{[\mu}\lambda_{\nu]}$; and the (symmetric) rate-of-strain tensor $\sigma_{\mu\nu} \equiv \lambda_{(\mu;\nu)} + a_{(\mu}\lambda_{\nu)}$. From the angular-velocity tensor can be defined an equivalent local angular-velocity vector, $\Omega^\mu$, by setting

$$\Omega^\mu = \tfrac{1}{2}(-g)^{-\frac{1}{2}}\epsilon^{\mu\nu\sigma\tau}\Omega_{\nu\sigma}\lambda_\tau. \qquad (C.29)$$

This can be solved for $\Omega_{\mu\nu}$,

$$\Omega_{\mu\nu} = (-g)^{-\frac{1}{2}}\epsilon_{\mu\nu\sigma\tau}\Omega^\sigma\lambda^\tau, \qquad (C.30)$$

which demonstrates the equivalence. All these canonical tensor quantities are projected into the local proper frame; that is,

$$a_\mu\lambda^\mu = \Omega_{\mu\nu}\lambda^\nu = \Omega_\mu\lambda^\mu = \sigma_{\mu\nu}\lambda^\nu \equiv 0. \qquad (C.31)$$

Now, we identify $\lambda^\mu \equiv {}_0\lambda^\mu$ and take the proper components of the canonical tensors with respect to the local tetrad, using Eq. (B.3) to introduce affinity components. Clearly, by Eq. (C.31), transvection with ${}_0\lambda^\mu$ itself will always give a zero result. For the acceleration vector, then, we have using Eq. (B.3) and Eq. (C.1),

$$a_\mu\, {}_a\lambda^\mu = {}_0\lambda_{\mu;\nu}\, {}_0\lambda^\nu\, {}_a\lambda^\mu = \Gamma_{00a} = a_a, \qquad (C.32)$$

so that our 3-vector **a** is precisely the local proper acceleration of the ${}_0\lambda^\mu$ congruence. Likewise from (C.2)

$$\Omega_\mu\, {}_a\lambda^\mu = \tfrac{1}{2}\epsilon_a^{\cdot bc}\Gamma_{cb0} = \Omega_a, \qquad (C.33)$$

and from (C.10)

$$\Omega_{\mu\nu}\, {}_a\lambda^\mu\, {}_b\lambda^\nu = -\Omega_{0ab} = \epsilon_{bac}\Omega^c, \qquad (C.34)$$

which identifies the 3-vector $\mathbf{\Omega}$ as the local angular velocity of the medium. Analyses by Synge,[5] Pirani,[10] and others have made it clear that, like **a**, this $\mathbf{\Omega}$ is an *absolute* entity: the angular velocity of the material medium with respect to Weyl's "compass of inertia."

The rate-of-strain tensor $\sigma_{\mu\nu}$ gives six proper components, all spacelike, and using (C.4),

$$\sigma_{\mu\nu}\, {}_a\lambda^\mu\, {}_b\lambda^\nu = -\Gamma_{(ab)0} = S_{ab}, \qquad (C.35)$$

so that S is the local, three-dimensional, rate-of-strain dyadic. With this, we have found transcriptions for all the canonical tensors and will turn to the interpretation of $\boldsymbol{\omega}$, Eq. (C.3).

Projecting the local time derivatives (i.e., the intrinsic derivatives in the ${}_0\lambda^\mu$ direction, for which we use the superimposed dot notation throughout) of the spacelike tetrad vectors themselves, one has for the timelike components

$${}_a\dot{\lambda}_\mu\, {}_0\lambda^\mu = -{}_a\lambda_\mu\, {}_0\dot{\lambda}^\mu = -{}_a\lambda_\mu\, a^\mu = -a_a \qquad (C.36)$$

from the orthogonality relations alone. And in the spatial directions, from Eq. (C.3) we have

$${}_a\dot{\lambda}_\mu\, {}_b\lambda^\mu = \Gamma_{0ab} = \epsilon_{abc}\omega^c. \qquad (C.37)$$

[9] See, for example, J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962).

[10] F. A. E. Pirani, Helv. Phys. Acta Suppl. IV, 198 (1956); Acta Phys. Polon. 15, 389 (1956).

The vector $\boldsymbol{\omega}$ thus describes the orthogonal propagation of the spacelike auxiliary triads *along* the ${}_0\lambda^\mu$ congruence; kinematically, $\boldsymbol{\omega}$ is the local angular velocity of the auxiliary orthonormal triad with respect to the compass of inertia. Conversely, $(-\boldsymbol{\omega})$ is the angular velocity of a "stable-platform" relative to the triad.

From Eqs. (C.36) and (C.37), we may express the condition for Fermi–Walker transport of the spacelike triad along a given line of the ${}_0\lambda^\mu$ congruence simply by setting $\boldsymbol{\omega} = 0$ on that line. Putting $\boldsymbol{\omega} = 0$ everywhere would prescribe the introduction of a tetrad field such that the spacelike triad attached to each material point represents a local, inertially nonrotating reference frame. It is an advantage of the dyadic notation that this condition is a 3-vector equation, form invariant under 3-space rotation. We show in Sec. C5 that it is always possible initially to introduce the tetrad field according to any such prescription for $\boldsymbol{\omega}$.

Elucidation of the kinematical significance of the quantities $\boldsymbol{\Omega}$, $\boldsymbol{\omega}$, and $\mathsf{S}$ is alternatively obtained by considering an equation for the proper orthogonal separation, say $\rho^\mu$, of two closely adjacent members of the ${}_0\lambda^\mu$ congruence. In the local, proper frame $\rho^\mu$ will appear as the displacement vector between two proximate material particles. Its rate of change with local time is given by[10]

$$\dot{\rho}^\mu = ({}_0\lambda^\mu_{\cdot;\nu} + {}_0\lambda^\mu a_\nu)\rho^\nu. \qquad \text{(C.38)}$$

Projecting $\rho^\mu$ onto the tetrad defines locally Cartesian spatial coordinates $r_a$, or components of a local displacement 3-vector $\mathbf{r}$, where

$$r_a = \rho^\mu\, {}_a\lambda_\mu. \qquad \text{(C.39)}$$

The local time derivative of these is found with the help of Eq. (C.38) and (C.11) to be given by

$$\dot{r}_a = 2\Omega_{a0b}r^b = [S_{ab} + \epsilon_{acb}(\Omega^c - \omega^c)]r^b. \qquad \text{(C.40)}$$

Equation (C.40) is valid to first order in the displacements $r_a$. These displacement components are a *Cartesian* vector, in the (flat) tangent space at the origin $r_a = 0$: the $S_{ab}$, $\Omega_c$ and $\omega_c$ are Cartesian components evaluated at $r_a = 0$. Remembering these limitations, we may still use dyadic notation:

$$\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r} = \mathsf{S}\cdot\mathbf{r} + \boldsymbol{\Omega}\times\mathbf{r} \qquad \text{(C.41)}$$

from which immediately

$$\tfrac{1}{2}\mathbf{D}\times\dot{\mathbf{r}} = \boldsymbol{\Omega} - \boldsymbol{\omega} \qquad \text{(C.42)}$$

and

$$\tfrac{1}{2}(\mathbf{D}\dot{\mathbf{r}} + \dot{\mathbf{r}}\mathbf{D}) = \mathsf{S}. \qquad \text{(C.43)}$$

These equations manifest the local kinematical significance of $\boldsymbol{\Omega}$, $\boldsymbol{\omega}$, and $\mathsf{S}$ and basically provide interpretations for the canonical tensors as well. Since Fermi–Walker transport of the basis vectors is accomplished by setting $\boldsymbol{\omega} = 0$, the interpretation of $\boldsymbol{\Omega}$ as the local angular velocity of the material relative to the compass of inertia is clear. In the general dyadic equations to be written later it is evident that a particularly convenient choice for $\boldsymbol{\omega}$ is rather to propagate the tetrads so that $\boldsymbol{\Omega} - \boldsymbol{\omega} = 0$. This alternative is called corotating transport, or "body-fixed axes," since as Eq. (C.42) shows, the local reference frame is thereby rotated with respect to the compass of inertia so as to follow the physical rotation of the neighboring members of the ${}_0\lambda^\mu$ congruence. Again, the condition for body-fixed axes is form invariant under 3-space rotation.

Interpretation of the quantities $\mathbf{L}$ and $\mathsf{N}$, which express characteristics of the auxiliary congruences, is somewhat less evident. In fact their significance, being more geometrical than physical, emerges most clearly in the special circumstance when the given timelike congruence comprises the orthogonal trajectories of a family of 3-surfaces immersed in space–time. This is discussed in some detail in the next section. First, however, the relationship of $\mathbf{L}$ and $\mathsf{N}$ to the properties of the spacelike congruences is obtained.

The first curvature vector of a curve of the congruence generated by ${}_a\lambda^\mu$ is defined by ${}_a\lambda_{\mu;\nu}\, {}_a\lambda^\nu$ ($a$ not summed), and its components in the local tetrad basis are

$$({}_a\lambda_{\mu;\nu}\, {}_a\lambda^\nu)_r\lambda^\mu = \Gamma_{aar} \quad (a \text{ not summed}). \qquad \text{(C.44)}$$

Referring to Eq. (C.4) we see that the timelike component is given by the diagonal element of $\mathsf{S}$,

$$\Gamma_{a0a} = S_{aa} \quad (a \text{ not summed}), \qquad \text{(C.45)}$$

which determines the rate of convergence in the ${}_a\lambda^\mu$ direction of the timelike congruence curves ${}_0\lambda^\mu$. For $r = b \neq a$ we have

$$\Gamma_{aab} = \epsilon_{dba}N^d_{\cdot a} + L_b \quad (a \text{ not summed}), \qquad \text{(C.46)}$$

which involves $\mathbf{L}$ and only the off-diagonal elements of $\mathsf{N}$. If we were to perform a 3-space rotation to diagonalize $\mathsf{N}$ at a given event, the spacelike components at that point of the first curvature vectors of the new set of auxiliary congruences thus obtained would be expressed by $\mathbf{L}$ alone. In general, of course, such a transformation does not diagonalize $\mathsf{N}$ elsewhere and it reappears in Eq. (C.46) at other events.

The geometrical meaning of the diagonal elements of $\mathsf{N}$ is more easily expressed in terms of the modified dyadic, $\mathsf{N} - \tfrac{1}{2}(\mathrm{tr}\ \mathsf{N})\mathsf{I}$. The $a$th diagonal element of this dyadic gives the rate of "twist" around the ${}_a\lambda^\mu$ direction applied to the triads in propagating them in the ${}_a\lambda^\mu$ direction itself. In a word, then, one might refer to these as the "torsions" of the spacelike congruence net.

### 5. Conditions on the Auxiliary Congruences. The ∇ Operator.

An orthonormal tetrad field aligned along a "given" congruence, generating three orthogonal but otherwise arbitrary auxiliary congruences, constitutes a complex geometrical structure. We wish, in this section, to remark about specializations of this auxiliary structure, some of which may be imposed in general, others only when the preferred congruence has special properties. While this discussion is not at all complete, it should at least show that the necessary equations for investigating such points are at hand in the dyadic notation. We first briefly discuss some specializations which are always available, then summarize several special cases which may occur, and finally, introduce the useful vector differential operator, $\nabla$, suggested by one such geometrical subcase.

The pertinent equations are, in fact, Eqs. (C.22), (C.24), and (C.25); for when an aligned but otherwise arbitrary tetrad field is initially introduced upon a given timelike congruence, the general orthogonal transformation $O_{ab}$ in these equations can often be selected to give a second, in some way special or canonical, tetrad field having the same alignment. The dyadic notation then allows the further generation (with constant $O_{ab}$) of a family of tetrad fields 3-space rotated from this second one, as was expounded previously.

The first example of this, encountered in the previous section, is the prescription of Fermi–Walker propagated axes *everywhere*, the condition $\boldsymbol{\omega} = 0$. *That this may be done in general* is clear from inspection of Eq. (C.22), when we regard the $\bar{\omega}_a$ as arbitrarily given initial fields, set $\omega_a = 0$, and solve for the three independent components of $\dot{O}_{ab}$ everywhere. A choice of $O_{ab}$ on one spacelike 3-surface then suffices to determine a solution. We thus demonstrate by direct construction a transformation leading to a new tetrad field with the desired property. Subsequent 3-space rotations (with $O_{ab}$ constant everywhere) clearly will preserve this property.

A second example is the imposition of body-fixed axes *everywhere*, $\boldsymbol{\Omega} - \boldsymbol{\omega} = 0$, the justification of which follows in exactly similar fashion.

Another important case is the imposition of the set of conditions $\mathbf{N} = 0$, $\mathbf{L} = 0$, $\boldsymbol{\omega} = 0$ *on a single world line* of the congruence. That this may be done follows again by construction of the required transformation. Given first $\bar{N}_{ab}$, $\bar{L}_a$, and $\bar{\omega}_a$, the 12 equations in (C.22), (C.24), and (C.25) can now be solved for the twelve partial derivatives of the three scalar fields in $O_{ab}$ *on the line*. With a choice of $O_{ab}$ at one point on the line it may by quadrature be suitably determined along and near the line to achieve any desired values of $\mathbf{N}$, $\mathbf{L}$, and $\boldsymbol{\omega}$.

An essential point is that while this last can always be done along a line or at a point, it cannot be done on manifolds of higher dimensions unless further integrability conditions are satisfied. Such conditions, however, introduce relations among the other 12 components of $\Gamma_{rst}$ (viz., $\mathbf{a}$, $\boldsymbol{\Omega}$, $\mathbf{S}$, referring to the timelike congruence) and so require the timelike congruence to have special properties. A typical situation occurs when one attempts simultaneously to impose Fermi–Walker propagation *everywhere* while also taking $\mathbf{N}$ and $\mathbf{L}$ to vanish on a line: the result is a constraint on the timelike congruence along that line.

We now proceed to summarize some similar cases in which partial degrees of integrability, or holonomity, are imposed on the congruence structure throughout space–time. The conditions take the form of the global vanishing of certain components of the object of anholonomity. The various conditions are not derived *ab initio* in the following; they are to be found for general spaces in Ref. 3. We are primarily interested here in specializing them to the case of a (3 + 1)-dimensional metric space with orthonormal tetrad vectors and then transscribing them into dyadic notation.

We consider first the geometrical situation in which one given pair of the four congruences is 2-forming. That is to say, the two congruences mesh together so as to form a (two-parameter) family of 2-surfaces embedded in the four-dimensional manifold. The condition for the $s$ congruence and the $t$ congruence to be 2-forming is

$$\Omega^{rst} = 0 \qquad (r \neq s, r \neq t). \tag{C.47}$$

(We emphasize again that these conditions are written for the case of orthonormal tetrads only.) For a given pair $(s, t)$ the inequalities allow only two values for the index $r$, and so two independent conditions result. There are six possible ways of pairing the congruences, and if we were to ask that *all* congruence pairs be 2-forming, we would require exactly one-half of the 24 independent components of the object of anholonomity to vanish everywhere. In dyadic terms from Eqs. (C.9)–(C.12) the 12 conditions given by Eq. (C.47) for this completely 2-forming case become:

$$\mathbf{\Omega} = 0, \qquad \boldsymbol{\omega} = 0,$$

$$S_{ab} = 0 \ (a \neq b), \quad N_{aa} = 0 \ (a \text{ not summed}), \qquad \text{(C.48)}$$

so that, in addition to the vanishing of the two angular velocities, $\mathsf{S}$ must be diagonal and $\mathsf{N}$ off-diagonal. The constraints on $\mathbf{\Omega}$ and $\mathsf{S}$ are of particular significance, since they restrict the physical congruences ${}_0\lambda^\mu$ for which this situation may exist.

We may consider, alternatively, the possibility that a given set of three congruences is 3-forming. This is here equivalent to the condition that the fourth congruence be 3-normal; that is, the unit vector generating this fourth congruence is everywhere proportional to the gradient of a scalar function, $\psi$, and so orthogonal to the family of 3-dimensional hypersurfaces, $\psi$ = constant, which essentially define a holonomic coordinate in the space. The condition for the $r$ congruence to be 3-normal is

$$\Omega^{rst} = 0 \qquad (s \neq r,\ t \neq r), \qquad \text{(C.49)}$$

which is very similar to (C.47) but differs in the effect of the inequalities. Here, when $r$ is given, $s$ and $t$ are allowed three values each, but the antisymmetry on $s$ and $t$ reduces the number of independent, nontrivial conditions to three. If, in this case, we ask that all four congruences be 3-normal, we again require the vanishing of 12 components of the object of anholonomity; clearly, in fact, the same 12 as for the case of complete 2-forming. The dyadic conditions for complete 3-normality, then, are already given by Eq. (C.48).

A large class of conditions, less restrictive than the complete cases covered by (C.48), could be considered. In accord with a dyadic approach however, which confers a special position exclusively on the timelike congruence, only those intermediate situations treating the three spacelike congruences impartially are of interest. There are four such subcases; the constraints for them follow immediately from Eq. (C.47) and (C.49) and they need only to be listed:

(1) All spacelike congruences are 2-forming with ${}_0\lambda^\mu$.

$$\mathbf{\Omega} - \boldsymbol{\omega} = 0, \qquad S_{ab} = 0 \quad (a \neq b). \qquad \text{(C.50)}$$

(2) All pairs of spacelike congruences are 2-forming.

$$\mathbf{\Omega} = 0, \qquad N_{aa} = 0 \quad (a \text{ not summed}). \qquad \text{(C.51)}$$

(3) All spacelike congruences are 3-normal.

$$\mathbf{\Omega} - \boldsymbol{\omega} = 0, \qquad S_{ab} = 0 \quad (a \neq b),$$

$$N_{aa} = 0 \quad (a \text{ not summed}). \qquad \text{(C.52)}$$

(4) The timelike congruence is 3-normal.

$$\mathbf{\Omega} = 0. \qquad \text{(C.53)}$$

It is worth noting that the 12 components of $\Omega^{rst}$ which are not concerned in *any* of the constraint equations presented in Eqs. (C.47)–(C.53) are $\mathbf{a}$, $\mathbf{L}$, the diagonal elements of $\mathsf{S}$, and the off-diagonal elements of $\mathsf{N}$. As we have brought out in previous discussions, these are precisely the 12 components of the first curvature vectors of the four congruences. The entire vanishing of the object of anholonomity is secured, then, by the requirements that all four congruences be 3-normal *and geodesic*. As we remarked in Sec. B, this would imply the vanishing of the Riemann tensor and the introduction of holonomic Minkowski coordinates.

In Case 4, Eq. (C.53), the separation of space and time is accomplished globally—space–time is a sandwich of spacelike 3-manifolds, each normal to the (everywhere nonrotating) timelike congruence. The Riemannian structure of space–time allows invariant measurements in any one of these 3-manifolds; it is, consequently, a Riemannian 3-manifold with an induced intrinsic metric and a second fundamental form (just $\mathsf{S}$) describing its immersion in the 4-space—the mathematics of this emerge naturally in Sec. D2. $\mathsf{N}$ and $\mathbf{L}$ now express exactly the nine components of the anholonomic affinity generated by an arbitrary triad field in a Riemannian 3-space. Even in the general case, this interpretation of $\mathsf{N}$ and $\mathbf{L}$ has much heuristic value, and completes our geometric discussion of these arrays.

If we pursue this last interpretation by introducing a vector operator $\nabla$ to denote triad-strangled three-dimensional covariant differentiation as in Eq. (B.11), e.g.,

$$\nabla_c M_{ab} = D_c M_{ab} + \Gamma^{\cdot\, d}_{c \cdot a} M_{db} + \Gamma^{\cdot\, d}_{c\cdot b} M_{ad}, \qquad \text{(C.54)}$$

we greatly simplify the notation in the dyadic differential equations to be presented in Sec. D. We denote $\nabla$ the *three-dimensional covariant differentiation* operator, although of course this interpretation is only immediately accessible geometrically in Case (4) (as differentiation in immersed subspaces). Without inquiring further here into the geometries of quotient subspaces, we merely regard the $\nabla$ operator in the general case as a useful notation. From the defining Eq. (C.54) we may calculate and tabulate the following useful formulas, where $\mathbf{V}$ is an arbitrary vector field, and $\mathsf{M}$ an arbitrary *symmetric* dyadic field:

$$\nabla \mathbf{V} = \mathbf{D}\mathbf{V} - [\mathsf{N} - \tfrac{1}{2}(\operatorname{tr} \mathsf{N})\mathsf{I} - \mathbf{L} \times \mathsf{I}] \times \mathbf{V}, \qquad \text{(C.55)}$$

$$\nabla\cdot\mathsf{V} = \mathbf{D}\cdot\mathsf{V} - 2\mathbf{L}\cdot\mathsf{V}, \tag{C.56}$$

$$\nabla\times\mathsf{V} = \mathbf{D}\times\mathsf{V} - \mathsf{N}\cdot\mathsf{V} - \mathbf{L}\times\mathsf{V}, \tag{C.57}$$

$$\begin{aligned}\nabla\times\mathsf{M} = {} & \mathbf{D}\times\mathsf{M} - \mathsf{M}\cdot\mathsf{N} - 2\mathsf{N}\cdot\mathsf{M} - \mathbf{L}\times\mathsf{M} \\ & + \mathbf{L}\cdot\mathsf{M}\times\mathsf{I} + \tfrac{1}{2}(\mathrm{tr}\,\mathsf{N})\mathsf{M} + (\mathrm{tr}\,\mathsf{M})\mathsf{N} \\ & + (\mathsf{N}:\mathsf{M})\mathsf{I} - \tfrac{1}{2}(\mathrm{tr}\,\mathsf{N})(\mathrm{tr}\,\mathsf{M})\mathsf{I},\end{aligned} \tag{C.58}$$

$$\begin{aligned}\nabla\times\mathsf{M} - \mathsf{M}\times\nabla = {} & \mathbf{D}\times\mathsf{M} - \mathsf{M}\times\mathbf{D} - 3\mathsf{M}\cdot\mathsf{N} \\ & - 3\mathsf{N}\cdot\mathsf{M} - \mathbf{L}\times\mathsf{M} + \mathsf{M}\times\mathbf{L} + (\mathrm{tr}\,\mathsf{N})\mathsf{M} \\ & + 2(\mathrm{tr}\,\mathsf{M})\mathsf{N} + 2(\mathsf{N}:\mathsf{M})\mathsf{I} - (\mathrm{tr}\,\mathsf{N})(\mathrm{tr}\,\mathsf{M})\mathsf{I},\end{aligned} \tag{C.59}$$

$$\nabla\cdot\mathsf{M} = \mathbf{D}\cdot\mathsf{M} - 3\mathbf{L}\cdot\mathsf{M} - \mathsf{N}\dot{\times}\mathsf{M} + (\mathrm{tr}\,\mathsf{M})\mathbf{L}. \tag{C.60}$$

## D. THE DYADIC PARTIAL DIFFERENTIAL EQUATIONS AND INTERPRETATION

### 1. The Dyadic Components of the Riemann Tensor

In this section we first introduce and discuss two alternate splittings of strangled components of the Riemann or curvature tensor into dyadic arrays.

Accordingly as they contain two, one, or no zeros, the strangled components of the symmetrized Riemann tensor in Eq. (B.8) may be gathered into four arrays with the property of covariance under 3-space rotation:

$$P_{ab} = \tfrac{1}{2}\epsilon_{acf}\epsilon_{bdg}S^{cdfg}, \tag{D.1}$$

$$Q_{ab} = 3S_{a00b}, \tag{D.2}$$

$$B_{ab} + \epsilon_{acb}t^{c} = \epsilon^{dc}_{\cdot\cdot b}S_{ad0c}. \tag{D.3}$$

We thus describe the 20 components of the curvature field of general relativity by three symmetric dyadics $\mathsf{P}$, $\mathsf{Q}$, $\mathsf{B}$ (the last is traceless) and a vector $\mathbf{t}$. In Sec. D2, when we write all the dyadic partial differential equations, we interpret $\mathsf{P}$ in terms of the intrinsic curvature of the spacelike 3-manifolds of a normal congruence. In Sec. D3 we derive several results allowing physical interpretation of the differential equations; in particular we there interpret $\mathsf{Q}$ as giving the tidal acceleration between neighboring test particles. An interpretation of $\mathsf{B}$ and $\mathbf{t}$ also appears in Sec. D3—they determine the differential (tidal) precession between neighboring (inertially oriented) test particles. It should be noted that, like $\mathsf{N}$, the dyadic $\mathsf{B}$ has a pseudocharacter under 3-space inversion.

The alternate splitting up of dyadic components of the Riemann tensor is suggested by considering the canonical resolution of this tensor, in four dimensions, into three irreducible tensorial parts with the same algebraic symmetries.[11] We write this in strangled form as

$$\begin{aligned}R_{rstu} = {} & C_{rstu} + (\eta_{r[u}H_{t]s} - \eta_{s[u}H_{t]r}) \\ & + \tfrac{1}{12}R(\eta_{r[u}\eta_{t]s} - \eta_{s[u}\eta_{t]r}),\end{aligned} \tag{D.4}$$

where

$$H_{rs} = R_{rs} - \tfrac{1}{4}R\eta_{rs}, \tag{D.5}$$

the strangled Ricci tensor is

$$R_{rs} = R^{t}_{\cdot rst}, \tag{D.6}$$

and its scalar contraction is the curvature scalar

$$R = R^{r}_{\cdot r}. \tag{D.7}$$

$C_{rstu}$ is the conformal curvature tensor (strangled) of Weyl; it is antidouble-dual; all its contractions are zero; it in general exists for Riemannian geometries in four or more dimensions, where its vanishing is the necessary and sufficient condition for the metric to be conformally flat. In four dimensions $C_{rstu}$ has ten independent components; upon resolution into proper dyadic arrays, according as the Lorentz indices contain one or two zeros, we obtain two symmetric dyadics (traceless, so having five components each) $\mathsf{A}$ and again the $\mathsf{B}$ of Eq. (D.3):

$$A_{ab} = C_{0a0b} = -\tfrac{1}{4}\epsilon_{acd}\epsilon_{bfg}C^{cdfg}, \tag{D.8}$$

$$B_{ab} = \tfrac{1}{2}\epsilon^{cd}_{\cdot\cdot b}C_{0acd}. \tag{D.9}$$

The dyadic $\mathsf{A}$, expressed in terms of the previous set, is one-half the traceless sum of $\mathsf{P}$ and $\mathsf{Q}$:

$$\mathsf{A} = \tfrac{1}{2}[\mathsf{P} + \mathsf{Q} - \tfrac{1}{3}(\mathrm{tr}\,\mathsf{P} + \mathrm{tr}\,\mathsf{Q})\mathsf{I}]. \tag{D.10}$$

To complete this alternate splitting, the ten components of the Ricci tensor may also be resolved into dyadic arrays. For physical reasons we prefer to introduce these from the strangled form of the Einstein tensor $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ which, in Einstein theory, is identified with the negative of the nongravitational stress-momentum-energy tensor, $T_{\mu\nu}$. [We have already adopted a unit of length such that the velocity of light $c = 1$; now we adopt a unit of mass such that the Newtonian constant of gravitation $\gamma$ is $(4\pi)^{-1}$.] In dyadic form we have then a symmetric stress dyadic $\mathsf{T}$, a momentum-density vector $\mathbf{t}$, and an energy-density scalar $\rho$:

$$T_{ab} = \tfrac{1}{2}R_{ab} - \tfrac{1}{4}R\eta_{ab}, \qquad t_a = \tfrac{1}{2}R_{a0}, \qquad \rho = -\tfrac{1}{2}R_{00} - \tfrac{1}{4}R. \tag{D.11}$$

The vector $\mathbf{t}$ was introduced previously in Eq. (D.3). The local proper system of a fluid is *defined* by the condition that ${}_0\lambda^{\mu}$ be an eigenvector of $T_{\mu\nu}$[12]:

[11] J. Géhéniau and R. Debever, Bull. Acad. Roy. Belg. Cl. Sci. **42**, 114, 252, 313, 608 (1956).

[12] J. L. Synge, Proc. London Math. Soc. **43**, 376 (1937).

$$T_{\mu\nu}\,{}_0\lambda^\nu = -\rho\,{}_0\lambda_\mu, \tag{D.12}$$

or simply

$$\mathsf{t} = 0. \tag{D.13}$$

In this proper system, $\rho$ is the proper energy or rest-mass density. The condition (D.13) is invariant under 3-space rotation. It is of especial importance in formulating many relativistic problems where the preferred congruence is of both kinematical and dynamical significance.

The dyadic $\mathsf{T}$ is, up to its trace, one-half the difference of $\mathsf{P}$ and $\mathsf{Q}$,

$$\mathsf{T} = \tfrac{1}{2}[-\mathsf{P} + \mathsf{Q} - (\mathrm{tr}\,\mathsf{Q})\mathsf{I}], \tag{D.14}$$

and $\rho$ is minus one-half the trace of $\mathsf{P}$. We note finally that the curvature scalar $R$ of Eq. (D.7) is given in terms of each set by

$$\tfrac{1}{2}R = -\mathrm{tr}\,\mathsf{T} - \rho = \mathrm{tr}\,\mathsf{P} + \mathrm{tr}\,\mathsf{Q}. \tag{D.15}$$

We have then two entirely equivalent sets of curvature dyadics—it is difficult to say which is to be preferred. In Einstein's theory the ten components of the Einstein tensor, $\mathsf{T}$, $\mathsf{t}$, and $\rho$, express the true (or non-self-excited) *sources* of the total gravitational curvature, and the ten components of conformal curvature, $\mathsf{A}$ and $\mathsf{B}$, express the expected ten components of a spin-2 gravitational *field*. From this point of view the second splitting is the more fundamental. Nevertheless the essential nonlinearity of Einsteinian theory appears both in the Bianchi Identities of Sec. D2, in 16 equations of which all these source and field terms are inextricably mixed, and again in the operational physical equations of test particle motion which are given in Sec. D3. In both of these, the more natural splitting of the Riemann tensor appears to be that first given, into the dyadics $\mathsf{P}$, $\mathsf{Q}$, $\mathsf{B}$, and $\mathsf{t}$, Eqs. (D.1)–(D.3).

The various possible radiative characters of Einsteinian gravitational fields are expressed, in close analogy with those of Maxwell fields, in the algebraically special forms of $C^{\mu}{}_{\cdot\nu\sigma\tau}$. The algebraic hierarchy for this due to Petrov, Pirani, and Sachs[13] leads, as might be expected, to simple canonical forms for our $\mathsf{A}$ and $\mathsf{B}$.

Summarizing this briefly, for a Type II field, the conform tensor has a singly degenerate principal null direction, which, strangled in any local proper frame, defines a unit 3-vector of propagation, say $\hat{\mathbf{v}}$; take this to be normal to a plane defined by otherwise arbitrary but orthogonal unit vectors $\hat{\mathbf{w}}$ and $\hat{\mathbf{u}}$; and then it may be shown that the field dyadics must be of the form

$$\begin{aligned}\mathsf{A} = (\tilde{a} - a)\hat{\mathbf{u}}\hat{\mathbf{u}} + (\tilde{a} + a)\hat{\mathbf{w}}\hat{\mathbf{w}} - 2\tilde{a}\hat{\mathbf{v}}\hat{\mathbf{v}} \\ + c(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}}) + b(\hat{\mathbf{v}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{v}}),\end{aligned} \tag{D.16}$$

$$\begin{aligned}\mathsf{B} = (\tilde{c} + c)\hat{\mathbf{u}}\hat{\mathbf{u}} + (\tilde{c} - c)\hat{\mathbf{w}}\hat{\mathbf{w}} - 2\tilde{c}\hat{\mathbf{v}}\hat{\mathbf{v}} \\ + a(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}}) + b(\hat{\mathbf{u}}\hat{\mathbf{v}} + \hat{\mathbf{v}}\hat{\mathbf{u}}).\end{aligned} \tag{D.17}$$

Here $a$, $\tilde{a}$, $c$, $\tilde{c}$, and $b$ are arbitrary scalars under 3-rotations. $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ are taken to form a right-handed orthonormal triad.

For a Type III algebraically special field the conform tensor has a doubly degenerate principal null direction—again denoting this by a unit $\hat{\mathbf{v}}$ we find that

$$\begin{aligned}\mathsf{A} = a(\hat{\mathbf{w}}\hat{\mathbf{w}} - \hat{\mathbf{u}}\hat{\mathbf{u}}) + c(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}}) \\ + b(\hat{\mathbf{v}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{v}}),\end{aligned} \tag{D.18}$$

$$\begin{aligned}\mathsf{B} = c(\hat{\mathbf{u}}\hat{\mathbf{u}} - \hat{\mathbf{w}}\hat{\mathbf{w}}) + a(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}}) \\ + b(\hat{\mathbf{u}}\hat{\mathbf{v}} + \hat{\mathbf{v}}\hat{\mathbf{u}}),\end{aligned} \tag{D.19}$$

which results from Eqs. (D.16), (D.17) on setting $\tilde{a} = \tilde{c} = 0$.

For a type-$N$ algebraically special field the conform tensor has but one principal null direction, triply degenerate, and the canonical forms simplify further ($b = a = 0$) to

$$\mathsf{A} = c(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}}), \tag{D.20}$$

$$\mathsf{B} = c(\hat{\mathbf{u}}\hat{\mathbf{u}} - \hat{\mathbf{w}}\hat{\mathbf{w}}). \tag{D.21}$$

The quadrupole character of this extreme far zone radiative gravitational field is nicely shown by these last forms, in conjunction with the test particle equations to be given in Sec. D3. Roy and Rhadakrishna[14] have obtained equivalent forms in a recent paper, together with elegant results for gravitational and electromagnetic–gravitational shock fronts. They characterize the type $N$ field, Eqs. (D.20)–(D.21), by saying that the 3-space quadrics associated with $\mathsf{A}$ and $\mathsf{B}$ are equal hyperbolic cylinders, coaxial (the $\hat{\mathbf{v}}$ direction!), with their other principal directions inclined at 45°. The scalar $c$ characterizes the gravitational field strength seen by an observer whose world line is ${}_0\lambda^\mu$; by itself, a type $N$ conform tensor has no nontrivial invariants. All of which is nicely analogous to the case of a null electromagnetic field.

[13] A. Z. Petrov, Sci. Trans. Kazan State University **114**, 55 (1954) [Translation by M. Karweit: Astron. Information, Trans. No. 29, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California (1963)]; F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957); R. K. Sachs, Z. Phys. **157**, 462 (1960).

[14] S. R. Roy and L. Radhakrishna, Proc. Roy. Soc. (London) **A275**, 245 (1963).

## 2. The Dyadic Partial Differential Equations

We now write the four sets of differential relations which must hold between our dyadic fields in full generality, the application and analysis of which are the essence of this dyadic formalism for general relativistic physics. These are, respectively, (a) the Differential Identities—16 equations (one scalar three vector, one dyadic) arising from Eq. (B.7) metric and curvature independent; (b) the Curvature Equations—20 equations (one vector, three dyadic, the first traceless) introducing the Riemann tensor components, from Eq. (B.8); (c) the Bianchi Identities—20 equations (three vector, two dyadic, the first traceless) relating the derivatives of the Riemann components, from the integrability conditions Eq. (B.13); and (d) the Commutation Formulas for anholonomic space and time differentiation, special cases of Eq. (B.12).

(a) Differential Identities

$$\nabla\cdot\mathbf{\Omega} = \mathbf{a}\cdot\mathbf{\Omega}, \tag{D.22}$$

$$\tfrac{1}{2}\nabla\times\mathbf{a} - (\dot{\mathbf{\Omega}} + \boldsymbol{\omega}\times\mathbf{\Omega}) = -\mathsf{S}\cdot\mathbf{\Omega} + (\mathrm{tr}\,\mathsf{S})\mathbf{\Omega}, \tag{D.23}$$

$$\nabla\cdot\mathsf{N} + \nabla\times\mathbf{L} = -2\mathbf{L}\cdot\mathsf{N} + (\mathrm{tr}\,\mathsf{N})\mathbf{L} - 2\mathsf{S}\cdot\mathbf{\Omega} + 2\boldsymbol{\omega}\times\mathbf{\Omega} \tag{D.24}$$

$$2\dot{\mathbf{L}} = (\nabla + \mathbf{a})\cdot[\mathsf{S}^{\star T} - (\mathrm{tr}\,\mathsf{S})\mathsf{I}] - \mathsf{S}^{\star}\dot{\times}\mathsf{N}^{\star}, \tag{D.25}$$

$$\begin{aligned}\dot{\mathsf{N}} - \tfrac{1}{2}(\mathrm{tr}\,\dot{\mathsf{N}})\mathsf{I} &= (\nabla + \mathbf{a})\cdot(\mathbf{\Omega} - \boldsymbol{\omega})\mathsf{I} \\ &\quad + \tfrac{1}{2}\mathsf{S}^{\star T}\times(\nabla + \mathbf{a}) - \tfrac{1}{2}(\nabla + \mathbf{a})\times\mathsf{S}^{\star} \\ &\quad - \tfrac{1}{2}\mathsf{S}^{\star}\cdot\mathsf{N}^{\star} - \tfrac{1}{2}\mathsf{N}^{\star T}\cdot\mathsf{S}^{\star T}.\end{aligned} \tag{D.26}$$

To shorten Eqs. (D.25)–(D.26) we have used the notation $\mathsf{S}^{\star} \equiv \mathsf{S} - (\mathbf{\Omega} - \boldsymbol{\omega})\times\mathsf{I}$ and $\mathsf{N}^{\star} \equiv \mathsf{N} - \frac{1}{2}(\mathrm{tr}\,\mathsf{N})\mathsf{I} - \mathbf{L}\times\mathsf{I}$. The superscript T denotes a transposed dyadic. The trace of Eq. (D.26) may be written in addition:

$$\begin{aligned}\mathrm{tr}\,\dot{\mathsf{N}} + 2\nabla\cdot(\mathbf{\Omega} - \boldsymbol{\omega}) &= 2\mathsf{N}:\mathsf{S} - (\mathrm{tr}\,\mathsf{N})(\mathrm{tr}\,\mathsf{S}) \\ &\quad - 2\mathbf{a}\cdot(\mathbf{\Omega} - \boldsymbol{\omega}) - 4\mathbf{L}\cdot(\mathbf{\Omega} - \boldsymbol{\omega}).\end{aligned} \tag{D.27}$$

The first two of these equations are remarkably simple, curvature-independent, general identities satisfied by the proper kinematic observables of any timelike congruence. The third, Eq. (D.24), expresses integrability conditions on the spatial parts, $\mathbf{L}$ and $\mathsf{N}$, of the anholonomic affinity. The remaining three relate the time derivatives of $\mathbf{L}$ and $\mathsf{N}$ to the properties of the preferred congruence.

(b) Curvature Equations

$$\nabla\cdot\mathsf{S} - \nabla(\mathrm{tr}\,\mathsf{S}) + \nabla\times\mathbf{\Omega} = 2\mathbf{\Omega}\times\mathbf{a} - 2\mathbf{t}, \tag{D.28}$$

$$\tfrac{1}{2}(\nabla\times\mathsf{S} - \mathsf{S}\times\nabla) - \tfrac{1}{2}(\nabla\mathbf{\Omega} + \mathbf{\Omega}\nabla) = \mathbf{a}\mathbf{\Omega} + \mathbf{\Omega}\mathbf{a} - \mathbf{a}\cdot\mathbf{\Omega}\mathsf{I} - \mathsf{B}, \tag{D.29}$$

$$\begin{aligned}\tfrac{1}{2}(\nabla\times\mathsf{N} - \mathsf{N}\times\nabla) - \tfrac{1}{2}(\nabla\mathbf{L} + \mathbf{L}\nabla) &= -\mathsf{N}\cdot\mathsf{N} + \tfrac{1}{2}(\mathrm{tr}\,\mathsf{N})\mathsf{N} - \mathbf{L}\mathbf{L} - [\tfrac{1}{8}(\mathrm{tr}\,\mathsf{N})^2 \\ &\quad - \tfrac{1}{4}\mathsf{N}:\mathsf{N} - \tfrac{1}{2}\mathbf{L}\cdot\mathbf{L}]\mathsf{I} + \mathsf{E} - \tfrac{1}{2}(\mathrm{tr}\,\mathsf{E})\mathsf{I},\end{aligned} \tag{D.30}$$

$$\dot{\mathsf{S}} + \boldsymbol{\omega}\times\mathsf{S} - \mathsf{S}\times\boldsymbol{\omega} - \tfrac{1}{2}(\nabla\mathbf{a} + \mathbf{a}\nabla) = -\mathsf{S}\cdot\mathsf{S} + \mathbf{a}\mathbf{a} - \mathbf{\Omega}\mathbf{\Omega} + (\mathbf{\Omega}\cdot\mathbf{\Omega})\mathsf{I} - \mathsf{Q}. \tag{D.31}$$

The traces of Eqs. (D.30) and (D.31) may be written in addition:

$$2\nabla\cdot\mathbf{L} = -\tfrac{1}{4}(\mathrm{tr}\,\mathsf{N})^2 + \tfrac{1}{2}\mathsf{N}:\mathsf{N} - \mathbf{L}\cdot\mathbf{L} + \mathrm{tr}\,\mathsf{E}, \tag{D.32}$$

$$\nabla\cdot\mathbf{a} - \mathrm{tr}\,\dot{\mathsf{S}} = \mathsf{S}:\mathsf{S} - \mathbf{a}\cdot\mathbf{a} - 2\mathbf{\Omega}\cdot\mathbf{\Omega} + \mathrm{tr}\,\mathsf{Q}. \tag{D.33}$$

Equation (D.30) may be referred to as the generalized equation of Gauss (c.f. Ref. 3, p. 278 and Ref. 4, p. 146). It contains only the spatial parts of the anholonomic affinity, $\mathbf{L}$ and $\mathsf{N}$, and the dyadic $\mathsf{E}$, defined as

$$\mathsf{E} \equiv -(\mathsf{P} + \tfrac{1}{2}\mathsf{S}\overset{\times}{\times}\mathsf{S} + \mathbf{\Omega}\mathbf{\Omega} + \boldsymbol{\omega}\mathbf{\Omega} + \mathbf{\Omega}\boldsymbol{\omega}). \tag{D.34}$$

In our case (4), when $\mathbf{\Omega} = 0$, the preferred congruence is 3-space normal, and Eq. (D.30) then comprises the six curvature equations for an imbedded Riemannian 3-space. The dyadic $\mathsf{E}$ reduces to

$$\mathsf{E} = -\mathsf{P} - \tfrac{1}{2}\mathsf{S}\overset{\times}{\times}\mathsf{S} \qquad (\mathbf{\Omega} = 0), \tag{D.35}$$

and is precisely the strangled Einstein 3-tensor for this imbedded space. The form explicitly reveals the dependence of the metric properties of the subspace on the four-dimensional curvature components $\mathsf{P}$ (which we have accordingly dubbed the induced curvature dyadic), and on the second fundamental form $\mathsf{S}$, the rate-of-strain of the timelike congruence. Upon taking the covariant divergence of Eq. (D.35), the dyadic equations may be used to show further that

$$\nabla\cdot\mathsf{E} = 0 \qquad (\mathbf{\Omega} = 0), \tag{D.36}$$

a vector equation expressing the three independent Bianchi Identities for a Riemannian 3-space. Finally, the scalar curvature of the subspace, $-2\,\mathrm{tr}\,\mathsf{E}$, is related to the spatial anholonomic affinity by Eq. (D.32).

Equations (D.28 and D.29) may together be referred to as the generalized equations of Codazzi (cf. Ref. 3, p. 278 and Ref. 4, p. 146) inasmuch as, again when $\mathbf{\Omega} = 0$, they are the usual eight partial differential equations for the second fundamental form of the imbedded 3-space. A special case of Eq. (D.28) in tensor form has been used by Rayner[15] in discussing Born-type rigid motions ($\mathsf{S} = 0$) in general relativity, (c.f. Ref. 2).

[15] C. B. Rayner, Compt. Rend. **248**, 929 (1959).

Equation (D.31) is essentially a kinematic relation for the preferred congruence; we return to its physical interpretation in Sec. D3. Its trace, Eq. (D.33), reduces for incoherent matter ($\mathsf{T} = 0$, $\mathbf{t} = 0$, $\mathbf{a} = 0$) to an equation whose tensor equivalent is found in Raychaudhuri's work.[16]

The quantities $\mathbf{L}$ and $\mathbf{N}$ do not appear explicitly in eighteen of the thirty-six equations, (D.22) to (D.33), althought they still play an implicit role in the "covariant" derivative, $\nabla$. It is often convenient to collect this particular set of equations in two nonsymmetric dyadic equations as follows:

$$\nabla\mathbf{a} - (\dot{\mathsf{S}} + \omega\times\mathsf{S} - \mathsf{S}\times\omega) + (\dot{\Omega} + \omega\times\Omega)\times\mathsf{I} = \mathsf{S}\cdot\mathsf{S} - \Omega\times\mathsf{S} - \mathsf{S}\times\Omega - \mathbf{a}\mathbf{a} + \Omega\Omega - (\Omega\cdot\Omega)\mathsf{I} + \mathsf{Q}, \qquad \text{(D.37)}$$

and

$$\nabla\Omega + \mathsf{S}\times\nabla = -2\mathbf{a}\Omega + (\mathbf{a}\cdot\Omega)\mathsf{I} + \mathsf{B} + \mathbf{t}\times\mathsf{I}. \qquad \text{(D.38)}$$

(c) Bianchi Equations

These follow from Eq. (B.13), but more directly can be obtained in dyadic form by differentiation of Eqs. (D.22)–(D.33), using the commutation formulas to be given in the following subsection.

$$\nabla\cdot\mathsf{Q} - \nabla(\operatorname{tr}\mathsf{Q}) - 2(\dot{\mathbf{t}} + \omega\times\mathbf{t}) = -\mathsf{S}\overset{\times}{\cdot}\mathsf{B} - 3\Omega\cdot\mathsf{B} - \Omega\times\mathbf{t} + 3\mathsf{S}\cdot\mathbf{t} + (\operatorname{tr}\mathsf{S})\mathbf{t} + \mathbf{a}\cdot[\mathsf{P} - \mathsf{Q} - (\operatorname{tr}\mathsf{P} - \operatorname{tr}\mathsf{Q})\mathsf{I}], \qquad \text{(D.39)}$$

$$\nabla\cdot\mathsf{B} - \nabla\times\mathbf{t} = \mathsf{S}\overset{\times}{\cdot}\mathsf{P} + 2\Omega\cdot\mathsf{Q} + \Omega\cdot\mathsf{P} - (\operatorname{tr}\mathsf{P})\Omega, \qquad \text{(D.40)}$$

$$\nabla\cdot\mathsf{P} = -\mathsf{S}\overset{\times}{\cdot}\mathsf{B} - 3\Omega\cdot\mathsf{B} - 3\Omega\times\mathbf{t} + \mathsf{S}\cdot\mathbf{t} - (\operatorname{tr}\mathsf{S})\mathbf{t}, \qquad \text{(D.41)}$$

$$\nabla\times\mathsf{Q} - \mathsf{Q}\times\nabla - 2(\dot{\mathsf{B}} + \omega\times\mathsf{B} - \mathsf{B}\times\omega) = (\mathsf{P} + \mathsf{Q})\times\mathbf{a} - \mathbf{a}\times(\mathsf{P} + \mathsf{Q}) - \mathbf{t}\times\mathsf{S} + \mathsf{S}\times\mathbf{t} + 3\Omega\mathbf{t} + 3\mathbf{t}\Omega - 2\Omega\cdot\mathbf{t}\mathsf{I} - \Omega\times\mathsf{B} + \mathsf{B}\times\Omega - 3\mathsf{S}\cdot\mathsf{B} - 3\mathsf{B}\cdot\mathsf{S} + 4(\operatorname{tr}\mathsf{S})\mathsf{B} + 2\mathsf{S}:\mathsf{B}\mathsf{I}, \qquad \text{(D.42)}$$

$$-\nabla\times\mathsf{B} + \mathsf{B}\times\nabla - \nabla\mathbf{t} - \mathbf{t}\nabla + 2\nabla\cdot\mathbf{t}\mathsf{I} - 2(\dot{\mathsf{P}} + \omega\times\mathsf{P} - \mathsf{P}\times\omega) = 2\mathbf{a}\times\mathsf{B} - 2\mathsf{B}\times\mathbf{a} + 2\mathbf{t}\mathbf{a} + 2\mathbf{a}\mathbf{t} - 4\mathbf{a}\cdot\mathbf{t}\mathsf{I} - \Omega\times\mathsf{P} + \mathsf{P}\times\Omega - \mathsf{P}\cdot\mathsf{S} - \mathsf{S}\cdot\mathsf{P} + 2(\operatorname{tr}\mathsf{S})\mathsf{P} - 2\mathsf{S}\overset{\times}{\times}\mathsf{Q}. \qquad \text{(D.43)}$$

The trace of Eq. (D.43) is of independent interest:

$$2\nabla\cdot\mathbf{t} - \operatorname{tr}\dot{\mathsf{P}} = -\mathsf{S}:\mathsf{P} + \mathsf{S}:\mathsf{Q} + (\operatorname{tr}\mathsf{S})(\operatorname{tr}\mathsf{P}) - (\operatorname{tr}\mathsf{S})(\operatorname{tr}\mathsf{Q}) - 4\mathbf{a}\cdot\mathbf{t}. \qquad \text{(D.44)}$$

[16] A. Raychaudhuri, Phys. Rev. 98, 1123 (1955).

The scalar Eq. (D.44) may be joined with a vector equation which is the difference of (D.39) and (D.41), to give four familiar equations for the stress dyadic $\mathsf{T}$, momentum density vector $\mathbf{t}$ and energy density $\rho$:

$$\nabla\cdot\mathbf{t} + [\dot{\rho} + (\operatorname{tr}\mathsf{S})\rho] = \mathsf{T}:\mathsf{S} - 2\mathbf{a}\cdot\mathbf{t}, \qquad \text{(D.45)}$$

$$\nabla\cdot\mathsf{T} - [\dot{\mathbf{t}} + \omega\times\mathbf{t} + (\operatorname{tr}\mathsf{S})\mathbf{t}] = \mathsf{S}\cdot\mathbf{t} + \Omega\times\mathbf{t} - \mathsf{T}\cdot\mathbf{a} + \rho\mathbf{a}. \qquad \text{(D.46)}$$

These are the "contracted Bianchi Identities" in dyadic form, commonly interpreted as conservation laws for energy and momentum.

(d) Commutation Formulas

A large variety of these may readily be inferred from Eq. (B.12). As was remarked, it is an inconvenience that neither the $\mathbf{D}$ nor $\nabla$ operator commutes with itself, or with time differentiation. We will give here only three which are of frequent occurrence in manipulating the intrinsic derivative operator $\mathbf{D}$; $\phi$ and $\mathbf{V}$ are arbitrary scalar and vector fields, respectively.

$$(\mathbf{D}\phi)^{\cdot} - \mathbf{D}(\dot{\phi}) = \mathbf{a}\dot{\phi} - \mathsf{S}\cdot\mathbf{D}\phi + (\Omega - \omega)\times\mathbf{D}\phi, \qquad \text{(D.47)}$$

$$\mathbf{D}\times\mathbf{D}\phi = 2\Omega\dot{\phi} + \mathbf{N}\cdot\mathbf{D}\phi + \mathbf{L}\times\mathbf{D}\phi, \qquad \text{(D.48)}$$

$$\mathbf{D}\cdot(\mathbf{D}\times\mathbf{V}) = 2\Omega\cdot\dot{\mathbf{V}} + \mathbf{N}:\mathbf{D}\mathbf{V} + \mathbf{L}\cdot\mathbf{D}\times\mathbf{V}. \qquad \text{(D.49)}$$

It is convenient however to give a quite complete tabulation of such formulas for the 3-space covariant operator $\nabla$; here $\mathsf{M}$ is an arbitrary symmetric dyadic. For the time–space commutation relations we have:

$$(\nabla\phi)^{\cdot} - \nabla(\dot{\phi}) = \mathbf{a}\dot{\phi} - \mathsf{S}^{\star}\cdot\nabla\phi, \qquad \text{(D.50)}$$

$$(\nabla\mathbf{V})^{\cdot} - \nabla(\;) = \mathbf{a}\dot{\mathbf{V}} - \mathsf{S}^{\star}\cdot\nabla\mathbf{V} - [\mathsf{S}^{\star T}\times(\nabla + \mathbf{a}) + (\nabla + \mathbf{a})\cdot(\Omega - \omega)\mathsf{I}]\times\mathbf{V}, \qquad \text{(D.51)}$$

$$(\nabla\times\mathsf{M})^{\cdot} - \nabla\times(\dot{\mathsf{M}}) = \mathbf{a}\times\dot{\mathsf{M}} - \mathsf{S}^{\star}\overset{\times}{\cdot}\nabla\mathsf{M} + [(\nabla + \mathbf{a})\times\mathsf{S}^{\star}]\cdot\mathsf{M} + \mathsf{M}\cdot[(\nabla + \mathbf{a})\times\mathsf{S}^{\star}] + [(\nabla + \mathbf{a})\times\mathsf{S}^{\star} - (\nabla + \mathbf{a})\cdot(\Omega - \omega)\mathsf{I}]\cdot[\mathsf{M} - (\operatorname{tr}\mathsf{M})\mathsf{I}] - [(\nabla + \mathbf{a})\times\mathsf{S}^{\star}]:\mathsf{M}\mathsf{I}. \qquad \text{(D.52)}$$

The analogous commutators for $(\nabla\cdot\mathbf{V})^{\cdot}$, $(\nabla\times\mathbf{V})^{\cdot}$, and $(\nabla\cdot\mathsf{M})^{\cdot}$ follow directly from Eqs. (D.51) and (D.52) by contraction and antisymmetrization, and so need not be exhibited. We have for convenience again introduced the nonsymmetric dyadic $\mathsf{S}^{\star}$ and its transpose $\mathsf{S}^{\star T}$:

$$\mathsf{S}^{\star} = \mathsf{S} - (\Omega - \omega)\times\mathsf{I},\quad \mathsf{S}^{\star T} = \mathsf{S} + (\Omega - \omega)\times\mathsf{I}. \qquad \text{(D.53)}$$

The commutation relations for spacelike directions are:

$$\nabla \times (\nabla \phi) = 2\mathbf{\Omega}\cdot\{\mathsf{I}\dot{\phi}\}, \tag{D.54}$$

$$\nabla\cdot(\nabla \times \mathbf{V}) = 2\mathbf{\Omega}\cdot\{\dot{\mathbf{V}} + \mathsf{S}^{\star}\cdot\mathbf{V}\}, \tag{D.55}$$

$$\nabla \times (\nabla \mathbf{V}) = -\mathsf{E}\times\mathbf{V} + 2\mathbf{\Omega}\cdot\{\mathsf{I}\dot{\mathbf{V}} - \tfrac{1}{2}\mathsf{S}^{\star}\times(\mathsf{I}\times\mathbf{V})\}, \tag{D.56}$$

$$\nabla\cdot(\nabla\times\mathsf{M}) = -\mathsf{E}\dot{\times}\mathsf{M} + 2\mathbf{\Omega}\cdot\{\dot{\mathsf{M}} + \tfrac{3}{2}\mathsf{S}^{\star}\cdot[\mathsf{M} - \tfrac{1}{3}(\mathrm{tr}\,\mathsf{M})\mathsf{I}]\}, \tag{D.57}$$

$$\nabla\cdot[\nabla\times(\mathsf{I}\times\mathbf{V})] = -\mathsf{E}\dot{\times}(\mathsf{I}\times\mathbf{V}) + 2\mathbf{\Omega}\cdot\{\mathsf{I}\times\dot{\mathbf{V}} + \tfrac{1}{2}\mathsf{S}^{\star}\cdot(\mathsf{I}\times\mathbf{V})\}. \tag{D.58}$$

These general relations appear quite complicated. Again, however, when $\mathbf{\Omega} = 0$ and the timelike congruence is 3-space normal, we discover simple, perspicuous equations. Equations (D.54) and (D.55) become the familiar vector identities; the rest reduce to dyadic forms of the Ricci identities in a Riemannian 3-space, with the Einstein dyadic $\mathsf{E}$ acting for the curvature tensor.

### 3. Physical Interpretations

Let us consider further the relative separation $\mathbf{r}$ of two closely adjacent particles of the ${}_0\lambda^{\mu}$ congruence, Eq. (C.41). This is a local Cartesian vector equation, correct to first order in $\mathbf{r}$; $\mathsf{S}$, $\mathbf{\Omega}$, and $\boldsymbol{\omega}$ are to be evaluated on one line of the congruence. Taking $\mathsf{N}$ and $\mathbf{L}$ to vanish on the line was tacitly necessary for interpretation of Eq. (C.40), for this condition implies that the spatial triad system is taken locally Cartesian and flat, and we in fact required this in order to write Eqs. (C.41)–(C.43), where the displacement $\mathbf{r}$ is a vector. We may thus say that Eq. (C.41) is not just pointwise valid, but rather is valid to first order in a flat metric 3-space carried along with the local observer. The observer is accelerating, and since we do not specialize $\boldsymbol{\omega}$ along the world line, his reference triad is arbitrarily rotating.

Differentiating Eq. (C.41) with respect to time, and substituting $\dot{\mathsf{S}}$ from Eq. (D.31) and $\dot{\mathbf{\Omega}}$ from Eq. (D.23), we can eliminate all such quantities relating to the whole congruence in favor of the local kinematic observables of one particle-observer (or of one line of the congruence with its reference tetrad), viz., $\mathbf{a}$ and $-\boldsymbol{\omega}$. These are respectively the vectorial reading of a linear accelerometer and the vector angular velocity of a (gyroscopically stabilized, or untorqued) "stable-platform."

We find as a result an equation for the observed spatial variation of $\mathbf{a}$:

$$\mathbf{a}_1 \equiv \mathbf{a} + \mathbf{r}\cdot\nabla\mathbf{a} = \mathbf{a}(1 - \mathbf{a}\cdot\mathbf{r}) + \ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \mathsf{Q}\cdot\mathbf{r}. \tag{D.59}$$

This is a quasi-Newtonian equation for $\mathbf{a}_1$, the accelerometer reading at the adjacent point $\mathbf{r}$, in terms of the accelerometer reading $\mathbf{a}$ at the origin of spatial coordinates and the relative acceleration $\ddot{\mathbf{r}}$. It is entirely written in local, proper or "operational" terms, and is immediately useful for the analysis of experiments. The usual centrifugal, Coriolis, and angular acceleration terms will be recognized. A special relativistic clock rate correction factor $(1 - \mathbf{a}\cdot\mathbf{r}/c^2)$—where $c^2 = 1$ in our units—is but another manifestation of the "red shift" predicted by special relativity for accelerating frames and recently verified in local terrestrial experiments using the Mössbauer effect (compare Ref. 5, p. 411).

The term $\mathsf{Q}\cdot\mathbf{r}$ is the general relativistic term expressing the tidal effect of the curvature tensor on the relative acceleration. When $\mathsf{Q}$ is written in terms of our second set of dyadics this term becomes

$$\mathsf{Q}\cdot\mathbf{r} = [\mathsf{A} + \mathsf{T} + \tfrac{1}{3}(\rho - 2\,\mathrm{tr}\,\mathsf{T})\mathsf{I}]\cdot\mathbf{r}. \tag{D.60}$$

In this form the contributions of the "source" and "field" parts of the Riemann tensor are separately revealed: for source-free regions one has just $\mathsf{A}\cdot\mathbf{r}$. If the test particles are free ($\mathbf{a} = \mathbf{a}_1 = 0$), Equation (D.59) reduces to the equation of geodesic deviation of Synge.[5] If on the other hand they are parts of a stress system obeying Hooke's law and the absolute accelerations $\mathbf{a}$, $\mathbf{a}_1$ are related to the stresses, one obtains the dynamical equations of Weber.[17] The dyadic partial differential equations, such as those for $\nabla\mathbf{a}$ and $\nabla\mathbf{\Omega}$, Eqs. (D.37) and (D.38), provide a generally valid instrument, expressed in an operational language, for the treatment of similar problems on the motion of macroscopic, continuous "test" bodies.

A similar equation may be found for the stable-platform angular velocity $-\boldsymbol{\omega}_1$, at $\mathbf{r}$, in terms of that at the origin, $-\boldsymbol{\omega}$. From Eqs. (D.28), (D.29), (D.25), and (D.26) and again (C.41), and setting $\mathsf{N} = 0$ and $\mathbf{L} = 0$, we obtain

$$-\boldsymbol{\omega}_1 \equiv -\boldsymbol{\omega} + \mathbf{r}\cdot\nabla(-\boldsymbol{\omega}) = (-\boldsymbol{\omega})(1 - \mathbf{a}\cdot\mathbf{r}) + \mathbf{a}\times(\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r}) - \mathsf{B}\cdot\mathbf{r} + \mathbf{t}\times\mathbf{r}. \tag{D.61}$$

Here all terms leading to a difference of $-\boldsymbol{\omega}_1$ and $-\boldsymbol{\omega}$ are nonclassical, of special or general relativistic origin. We again find a clock rate correction factor. The second special relativistic term is the differential Thomas precession. These two terms combined can be derived from the usual Thomas precession formula, in the differential limit, if care is taken to express all precession rates in terms of local proper

[17] J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961), Chap. 8.

times. In the last two terms we again note separate contributions from the field and source parts of the Riemann tensor: a "spin" term $-\mathsf{B}\cdot\mathbf{r}$, arising from the conformal tensor, and an "orbital" term $\mathbf{t}\times\mathbf{r}$, from the Einstein tensor. For geodesic observers, only these general relativistic terms will remain; they may be denoted the differential Fokker precession.[18]

Equations (D.59) and (D.61) show how in principle the fourteen Riemann components $\mathsf{Q}$, $\mathsf{B}$, and $\mathbf{t}$ may be experimentally determined from local differential kinematical measurements near, and on, one arbitrarily given timelike world line. As was remarked previously, the remaining six components, in the induced curvature dyadic $\mathsf{P}$, are in principle determinable from local spatial surveying in a triad system, Eq. (D.30); this means that their geometric effects will be second order in the spatial displacement components $r_a$. An experimental approach to the measurement of $\mathsf{P}$ would no doubt instead involve kinematical experiments on $\mathsf{Q}$, $\mathsf{B}$, and $\mathbf{t}$ as above, but made by two or more point-observers in rapid relative motion. These complications will not arise in source-free regions, however: for expressing $\mathsf{P}$ by

$$\mathsf{P} = \mathsf{A} - \mathsf{T} + \tfrac{1}{3}(\mathrm{tr}\,\mathsf{T} - 2\rho)\mathsf{I} \qquad \text{(D.62)}$$

and recalling Eq. (D.60), we clearly have in this case $\mathsf{P} = \mathsf{Q} = \mathsf{A}$.

As a final illustration we obtain an equation for the quasi-Newtonian "gravitational field" of a nonrotating ($\mathbf{\Omega} = 0$) static distribution of matter with proper energy density $\rho$ and stress dyadic $\mathsf{T}$. The matter is represented by a congruence ${}_0\lambda^\mu$ defined by the condition $\mathbf{t} = 0$, and everywhere nonrotating ($\boldsymbol{\omega} = 0$) auxiliary triads are introduced. A static distribution is defined operationally by the condition that in this tetrad system the local time derivative of every kinematic observable must vanish. We, of course, already have $\dot{\mathbf{\Omega}} = \dot{\boldsymbol{\omega}} = 0$, but specifically impose the further conditions $\dot{\mathbf{a}} = 0$ and $\mathsf{S} = 0$, the latter being required to ensure that all relative displacements $\mathbf{r}$ are time independent.

When all these conditions ($\mathbf{\Omega} = \boldsymbol{\omega} = \mathbf{t} = \dot{\mathbf{a}} = \mathsf{S} = 0$) are invoked, Eqs. (D.25) and (D.26) show that $\dot{\mathsf{L}} = \dot{\mathsf{N}} = 0$, and the other dyadic equations then directly yield the same result for the local time derivative of every remaining quantity. For instance, the scalar Bianchi identity Eq. (D.45) has the immediate consequence, $\dot{\rho} = 0$.

We now imagine a population of proper Newtonian observers, each of whom prefers to ascribe his kinematic observations not to his own absolute acceleration $\mathbf{a}$, but rather to a "gravitational field of force" with intensity $\mathbf{F} = -\mathbf{a}$. The "gravitational field equation" is then just Eq. (D.33) which, under the imposed conditions, may be written

$$\nabla\cdot\mathbf{F} = -4\pi\gamma\left(\rho_M - \frac{1}{c^2}\,\mathrm{tr}\,\mathsf{T}\right) + \frac{1}{c^2}\,\mathbf{F}\cdot\mathbf{F}, \qquad \text{(D.63)}$$

where we have put $-\mathbf{F}$ for $\mathbf{a}$; substituted for tr $\mathsf{Q}$ its equivalent, $\rho$ − tr $\mathsf{T}$; restored dimensional factors; and defined a proper mass density, $\rho_M \equiv \rho/c^2$.

When $\mathbf{\Omega} = 0$ it follows from Eq. (D.23) that $\nabla\times\mathbf{a} = 0$, and this, together with $\dot{\mathbf{a}} = 0$, is sufficient to permit expressing $\mathbf{F}$ as the gradient of a time-independent scalar:

$$\mathbf{F} = -\nabla\phi, \qquad \dot{\phi} = 0. \qquad \text{(D.64)}$$

Equation (D.63) will then take the form

$$\nabla^2\phi = 4\pi\gamma\left(\rho_M - \frac{1}{c^2}\,\mathrm{tr}\,\mathsf{T}\right) - \frac{1}{c^2}\,(\nabla\phi)^2. \qquad \text{(D.65)}$$

For the prescribed conditions this is an exact equation reducing to Poisson's equation in the nonrelativistic approximation. If we also rewrite Eq. (D.37) in these terms and for these conditions, we find the following expressions for the tidal acceleration dyadic $\mathsf{Q}$:

$$\mathsf{Q} = -\nabla\mathbf{F} + \frac{1}{c^2}\,\mathbf{F}\mathbf{F}$$
$$= \nabla\nabla\phi + \frac{1}{c^2}\,(\nabla\phi)(\nabla\phi). \qquad \text{(D.66)}$$

*Note added in proof*: In a private communication, Dr. F. A. E. Pirani has very kindly called our attention to the "method of projection" of Carlo Cattaneo.[19] We were completely unaware of this work, whose relation to the present formulation should be noted. Our operator $\nabla$, denoted by us the operator of "spatial covariant differentiation," is precisely the covariant operator of "transverse differentiation" of Cattaneo, strangled. Those of our equations such as (D.37) and (D.38) not explicitly involving $\mathsf{N}$ and $\mathsf{L}$ can of course be immediately "unstrangled" by multiplication with ${}_a\lambda^\mu$, ${}_b\lambda^\nu$, etc., to give covariant equations not depending on a choice of auxiliary congruences; such equations are thus derivable by the method of projection. On the other hand, our equations (D.24), (D.25), (D.26), and (D.30) explicitly contain $\mathsf{N}$ and $\mathsf{L}$, and seem to be much less accessible in covariant language, while vital for the completeness of the total set.

[18] A. D. Fokker, Proc. Roy. Acad. (Amsterdam) **23**, 729 (1920).

[19] See, for example, C. Cattaneo, Compt. Rend. **248**, 197 (1959); I. Cattaneo-Gasparini, Compt. Rend. **252**, 3722 (1961).